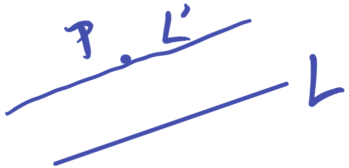
Geometry, Topology, and Group Theory

Last time: We've learned so much!

Certain groups can/cannot act (geometrically) on the same graph/space.

Today: There's so much more to learn!

Hyperbolic Geometry



Euclid's Postulates 1-4 boring.

5 Given a point P not on line L

$\exists!$ line L' through P & not intersect L.

Lobachevsky/Poincaré: There is geometry without 5

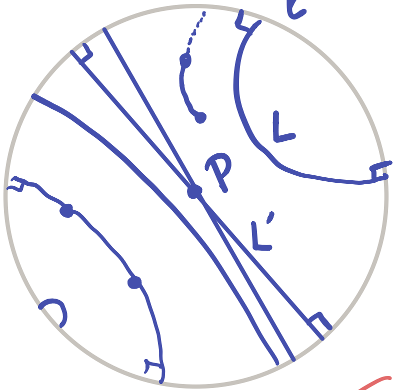
$\rightsquigarrow$ Hyperbolic plane.

Hyperbolic Plane $\mathbb{H}^2$

open disk.

Defn 1

Compare Farey Graph.

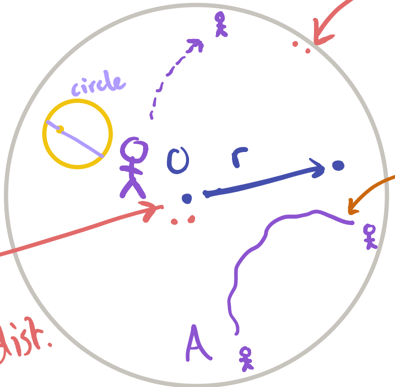


The straight lines are pieces of circles/lines $\perp$ to boundary. $\Longrightarrow$ metric is a multiple of one below

Riemannian geometry

really far apart.

Defn 2.



distances almost same as Eucl. dist.

geodesic

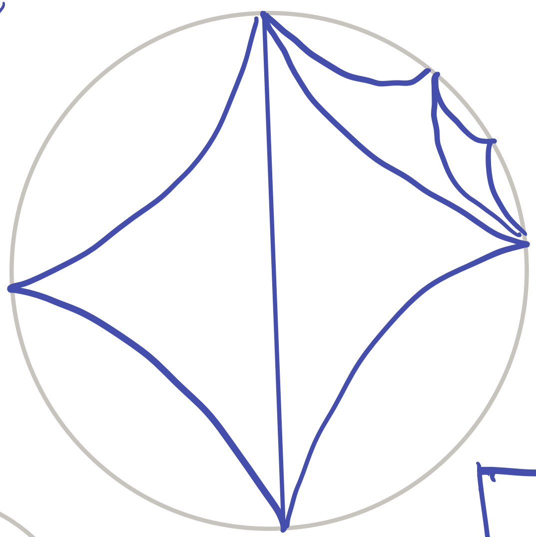
$\downarrow$

compare BS(1,2)

Metric: $\dfrac{\text{Euclidean metric}}{(1-r^2)}$

$\Longrightarrow$ straight lines as above.

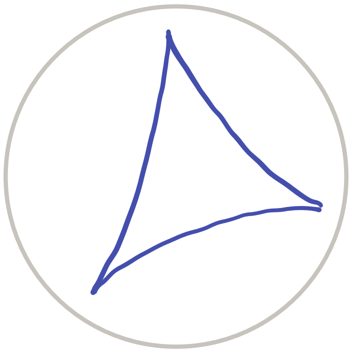
Farey graph



all triangles congruent in $\mathbb{H}^2$

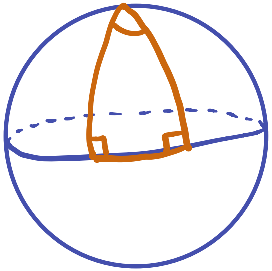
all have interior angles 0

(all triangles "skinny")



sum of interior angles $< \pi$.

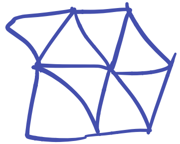
Compare spherical geometry



interior angles $> \pi$

Which groups act on $\mathbb{H}^2$?

For $\mathbb{E}^2$ have reflection groups, e.g. W_{333}



and $\mathbb{Z}^2$

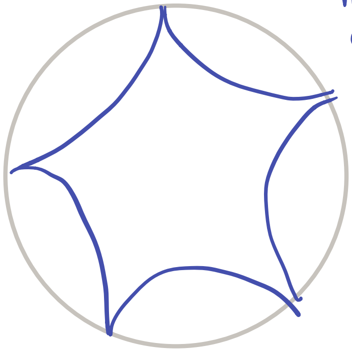
all of these coming from tilings

Let's look for tilings of $\mathbb{H}^2$.

Looking for tiles in $\mathbb{H}^2$

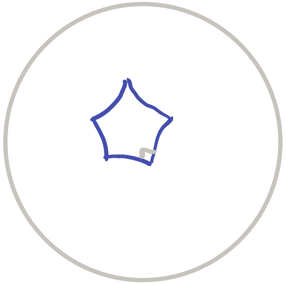
int. $\measuredangle$'s $\sim 3\pi/5$

small n-gons have nearly Euclidean interior angle sums $\pi(n-2)$



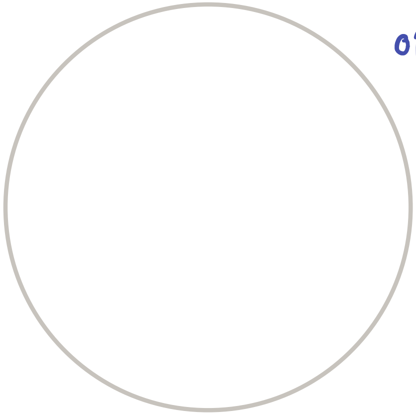
interior angles 0

IVT $\Rightarrow \exists$ regular right angled pentagon!



Now tile!

Aside: Defn #3 of $\mathbb{H}^2$.

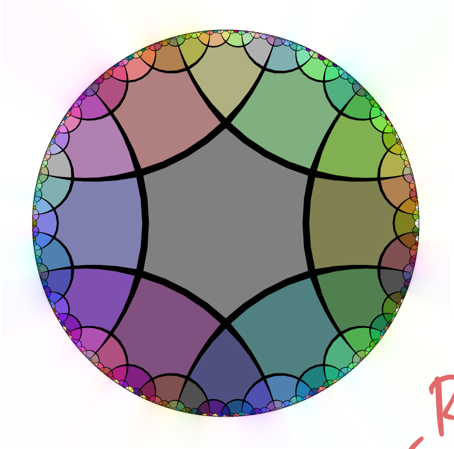


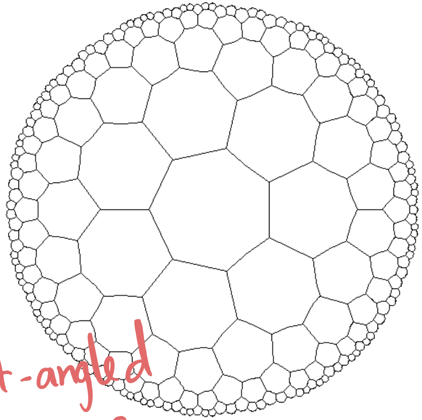
open unit disk

Isometries are:

$$\left\{ \begin{array}{c}\text{Möbius transformation} \\ \text{preserving open unit disk}\end{array} \right\}$$

$$\longleftrightarrow \left\{ f(z) = \frac{az+b}{cz+d} : \begin{array}{c} a,b,c,d \\ \in \mathbb{R} \end{array} \right\}$$





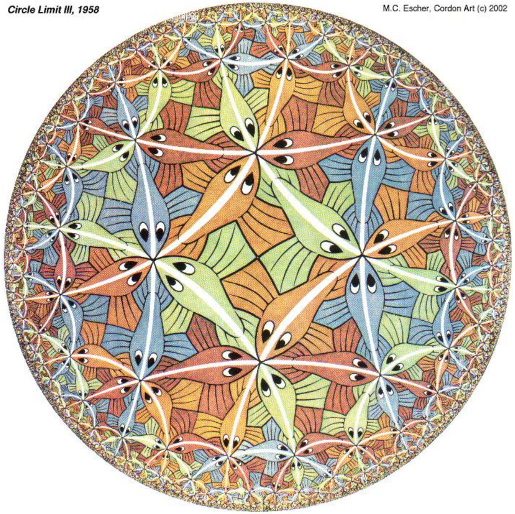
Right-angled
Coxeter/reflection gps

reflection group

$$\langle x_1,\dots,x_5 : (x_1x_2)^2=(x_2x_3)^2=\dots=(x_5x_1)^2=\mathrm{id}\rangle$$

Now have many new gps, not QI to Euclidean gps
W_{333} etc.

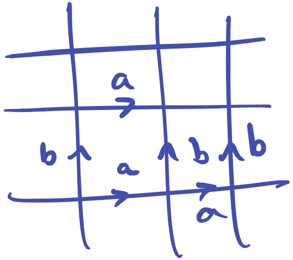
Circle Limit III, 1958

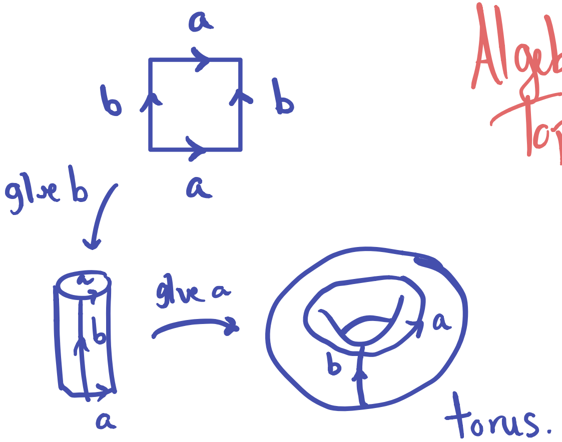


Connection to Topology

Algebraic Topology.

$\mathbb{E}^2$





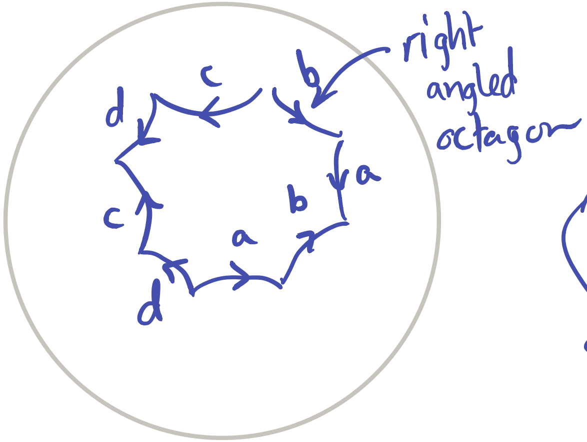
the loop around the square is the relation

$\langle a, b : aba^{-1}b^{-1} = id \rangle$

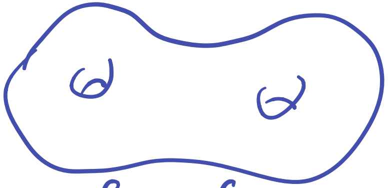
$\cong \mathbb{Z}^2$

QI to $\mathbb{E}^2$

$\mathbb{H}^2$



S_2



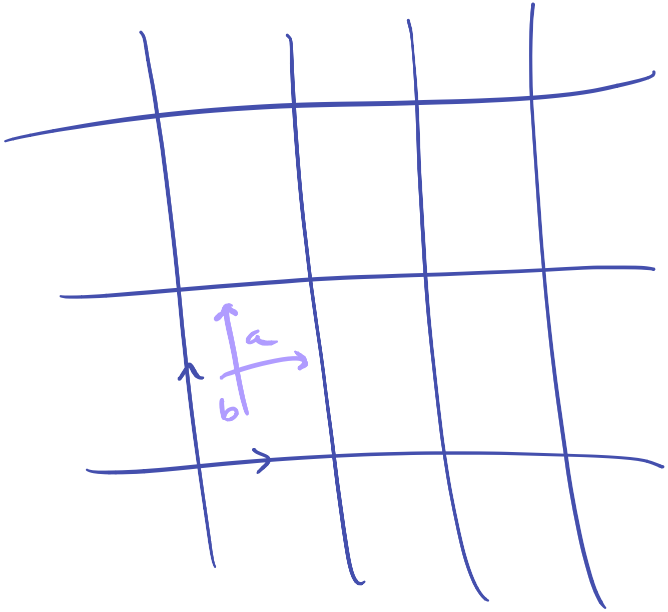
Surface of genus 2.

$\langle a, b, c, d :$

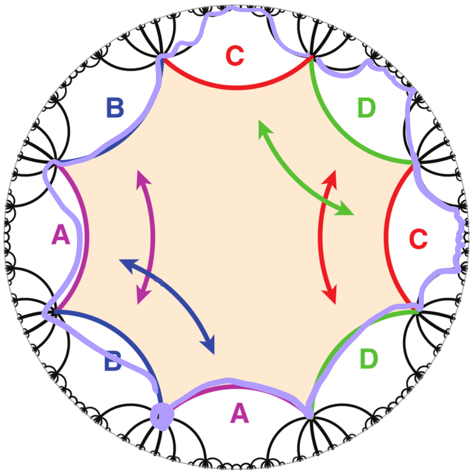
$aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$

fundamental gp of S_2

QI to $\mathbb{H}^2$



$\mathbb{E}^2$

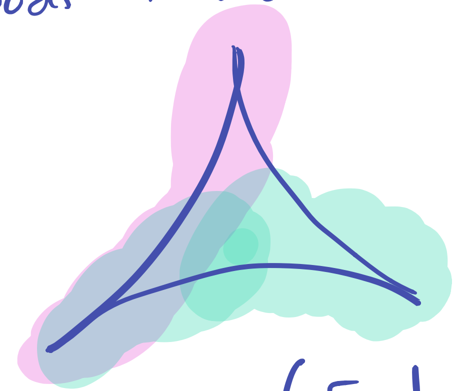


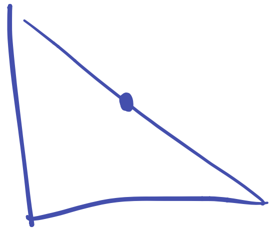
$\mathbb{H}^2$

Milnor-Schwarz:
fund. gp of $S_2 \underset{QI}{\sim} \mathbb{H}^2$

Hyperbolic Groups à la Gromov

A space is δ-hyperbolic if for any triangle, the δ-neighborhoods of two sides together contain the 3^{rd} side.





Facts. • $\mathbb{H}^2$ is δ-hyperbolic $(\delta = \log 2\,?)$
• δ-hyp. is a QI invt $\Rightarrow$ fund gp of S_2 is δ-hyp.

Two Theorems of Gromov

Thm. Most groups are δ-hyperbolic.

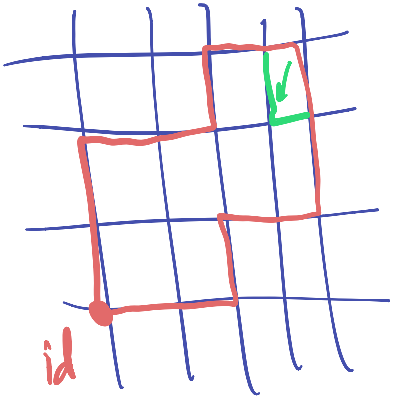
Thm. A group is δ-hyp $\iff$ its word problem is solvable in linear time.

Geometry

Group theory

Why does fund gp of S_2 have linear time soln to WP?

$\langle a,b,c,d : aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$

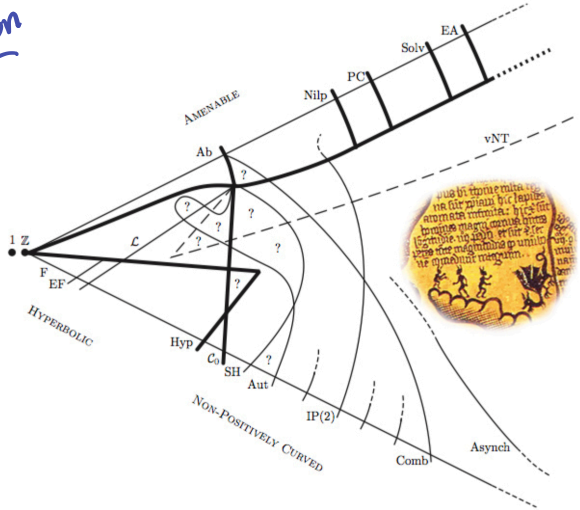


Any closed loop in Cayley graph must use ≥ 6 sides of a single octagon

So can replace word of length 6 with word of length 2

SHORTENING.

Bridson



Here there be dragons.

Key: Ab — abelian, Nilp — nilpotent, PC — polycyclic, Solv — solvable, EA — elementary amenable, F = free, EF — elementarily free, $\mathcal{L}$ — limit, Hyp — hyperbolic, $\mathcal{C}_0$ — CAT(0), SH — semi-hyperbolic, Aut — automatic, IP(2) — quadratic isoperimetric inequality, Comb — combable, Asynch — asynchronously combable, vNT — the von Neumann–Tits line. The question marks indicate regions for which it is unknown whether any groups are present.