Geometry, Topology, and Group Theory

Last time: We've learned so much!

Certain groups can/cannot act (geometrically) on the same graph/space.

Today: There's so much more to learn!
Hyperbolic Geometry

Euclid's Postulates (1-4) boring.
5) Given a point $P$ not on line $L$, 
exists line $L'$ through $P$ & not intersect $L$.

Lobachevsky/Poincaré: There is geometry without 5

→ Hyperbolic plane
Hyperbolic Plane $\mathbb{H}^2$

**Defn 1**

Compare Farey Graph.

**Defn 2**

Distances almost same as Eucl. dist.

The straight lines are pieces of circles/lines $\perp$ to boundary. $\Rightarrow$ metric is a multiple of one below.

Riemannian geometry

Metric:

$$\text{Euclidean metric}$$

$$1 - r^2$$

$\Rightarrow$ straight lines as above.
all triangles congruent in \( \mathbb{H}^2 \)
all have interior angles 0
(all triangles “skinny”)

sum of interior angles < \( \pi \).

Compare spherical geometry
interior angles > \( \pi \).
Which groups act on $1H^2$?

For $E^2$ have reflection groups, e.g. $W_{333}$ and $Z^2$ all of these coming from tilings.

Let's look for tilings of $1H^2$. 
Looking for tiles in $\mathbb{H}^2$

small $n$-gons have nearly Euclidean interior angle sums $\pi(n-2)$

$1V \Rightarrow$ regular right angled pentagon!

Now tile!
Aside: Defn #3 of $\mathbb{H}^2$.

Isometries are:

\[ \{ \text{Möbius transformation preserving open unit disk} \} \]

\[ \longleftrightarrow \{ f(z) = \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R} \} \]
Now have many newgps, not QT to Euclidean gps W333 etc.
Connection to Topology

\[ \mathbb{H}^2 \]

\[ \begin{array}{c}
\text{glue } b \\
\text{glue } a
\end{array} \]

\[ \langle a, b : aba^{-1}b' = id \rangle \approx \mathbb{Z}^2 \]

\[ \text{torus. QI to } \mathbb{H}^2 \]

\[ \langle a, b, c, d : \text{right angled octagon} \rangle \]

\[ \text{Surface of genus 2.} \]

\[ \langle a, b, c, d : \text{fundamental gp of } S_2 \rangle \]

QI to \( H^2 \)
\[ \mathbb{H}^2 \]

Milnor-Schnirelman:
fund. gp of \( S_2 \rightharpoonup_\mathbb{H}^2 \)

\[ \mathbb{H}^2 \]
Hyperbolic Groups à la Gromov

A space is $\delta$-hyperbolic if for any triangle, the $\delta$-neighborhoods of two sides together contain the 3rd side.

Facts:
- $H^2$ is $\delta$-hyperbolic ($\delta = \log 2$?)
- $\delta$-hyp. is a QI invt $\Rightarrow$ fundgp of $S_2$ is $\delta$-hyp.
Two Theorems of Gromov

Thm. Most groups are hyperbolic.

Thm. A group is $\delta$-hyp $\Leftrightarrow$ its word problem is solvable in linear time.
Why does fund gp of $S_2$ have linear time soln to WP?

\[ \langle a, b, c, d : \text{aba} = \text{b-c-d-c-d-d} \rangle \]

Any closed loop in Cayley graph must use $\geq 6$ sides of a single octagon.

So can replace word of length 6 with word of length 2.

SHORTENING.
Bridgon

Here there be dragons.

Key: Ab — abelian, Nilp — nilpotent, PC — polycyclic, Solv — solvable, EA — elementary amenable, F = free, EF — elementarily free, L — limit, Hyp — hyperbolic, C_0 — CAT(0), SH — semi-hyperbolic, Aut — automatic, IP(2) — quadratic isoperimetric inequality, Comb — combable, Asynch — asynchronously combable, vNT — the von Neumann–Tits line. The question marks indicate regions for which it is unknown whether any groups are present.