ANNOUNCEMENTS FEB 2

- Cameras on
- HW 2 due Thu 3:30
- Groups/topics due Feb 5
- OH Fri 2-3, appt
- Way-too-early course feedback Canvas → Quizzes

Cayley graphs

G = group

S = genset

$\Gamma_{G,S}$ graph

vertices: G

edges: $g \xrightarrow{s} gs$

$\forall$ $g \in G$
$s \in S$.

Last time: $G \curvearrowright \Gamma_{G,S}$

$$G \times V(\Gamma_{G,S}) \longrightarrow V(\Gamma_{G,S})$$

$$g \cdot h = gh.$$

gp elt   vertex

This rule also tells you where edges go.

$$\left(h \xrightarrow{s} hs \right) \xrightarrow{g} \left(gh \xrightarrow{s} ghs \right)$$

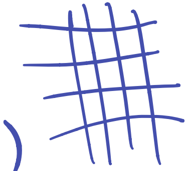
preserving labels/arrows.

Also: $G \hookrightarrow \mathrm{Sym}^+(\Gamma_{G,S})$

Thm. The natural map

$$\Phi : G \longrightarrow \mathrm{Sym}^+(\Gamma_{G,S})$$

defined above is an isomorphism.



Pf. Remains to show surjectivity.

Let $\overset{\alpha}{\cancel{\Phi}} \in \mathrm{Sym}^+(\Gamma_{G,S})$

Need: $\overset{\alpha}{\cancel{\Phi}} = \Phi(g)$ some $g \in G$.

Which g? Take "$g = \underset{\alpha}{\cancel{\Phi}}(e)$"

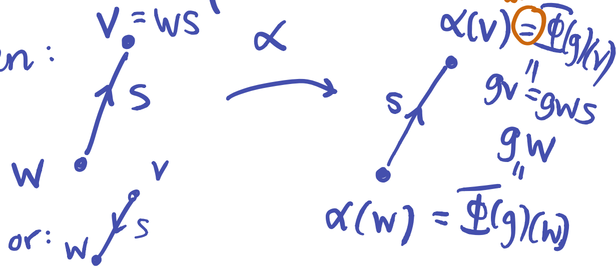
So $\overset{\alpha}{\cancel{\Phi}}$ & $\Phi(g)$ agree on the vertex e.

Induct on distance from e: we'll show $\Phi(g)$ & α agree on all vertices of distance n from e.

Base case: distance 0

Inductive step: Assume $\Phi(g)$, α agree on vertices of distance n from e. Say v has distance $n+1$

Then:



distance n from e.

From Meier:

$G \circlearrowright X$ = top space.

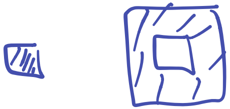
(example: $\mathbb{Z}^2 \circlearrowright \mathbb{R}^2$)

A <u>fundamental domain</u> for the action is a subset $\mathcal{F} \subseteq X$

s.t. ① $\mathcal{F}$ closed ③ connected

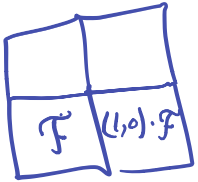
② $\bigcup_{g \in G} g \cdot \mathcal{F} = X$

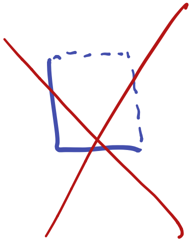
and no proper subset of $\mathcal{F}$ satisfies ① & ②. & ③



In the example can take

$\mathcal{F}$ = unit square

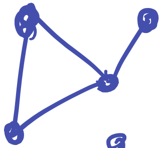




Issue #1. We don't know what closed subsets of a graph are.

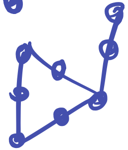
Lots of fundamental domains:

For us: $G \curvearrowright \Gamma$ = graph.



Γ

Γ' = (barycentric) subdivision of Γ (subdivide all edges of Γ).



Γ'

A fundamental domain for $G \curvearrowright \Gamma$ is a subgraph $\mathcal{F} \subseteq \Gamma'$ s.t.

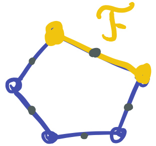
① $\mathcal{F}$ connected.

② $\bigcup_{g \in G} g \cdot \mathcal{F} = \Gamma'$

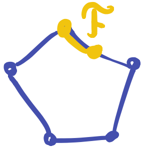
& $\mathcal{F}$ minimal with respect to ① & ②.

Examples

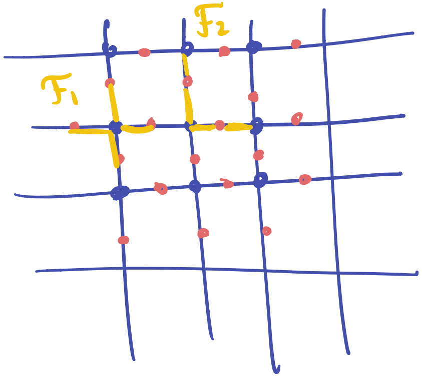
① $\mathbb{Z}/5 \curvearrowright$



② $D_5 \curvearrowright$



③ $\mathbb{Z}^2 \curvearrowright$



Thm. Say $G \circlearrowright \Gamma$ ← connected

& $F \subseteq \Gamma'$, connected. subgraph

& $\bigcup_{g \in G} g \cdot F = \Gamma \, (= \Gamma')$

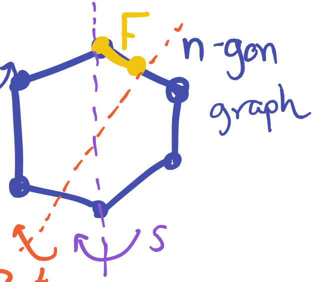
(e.g. F = fund. domain)

Let $S = \{ g \in G : g \cdot F \cap F \neq \emptyset \}$

Then S generates G.

The smaller F is, the smaller S is.

That's why we care about fundamental domains.

Example. $D_n \circlearrowright$ 

$S = \{s, t\}$

Q. Is G acts faithfully and F is a fund domain, is S minimal?

(Tolson).

Example next time:

$\mathrm{Sym}_n \circlearrowright K_n$.

Thm. Say $G \curvearrowright \Gamma$ ← connected

& $F \subseteq \Gamma'$ connected. subgraph

& $\bigcup_{g\in G} g\cdot F = \Gamma \, (= \Gamma')$

(e.g. F = fund. domain)

Let $S = \{g \in G : g\cdot F \cap F \neq \emptyset\}$

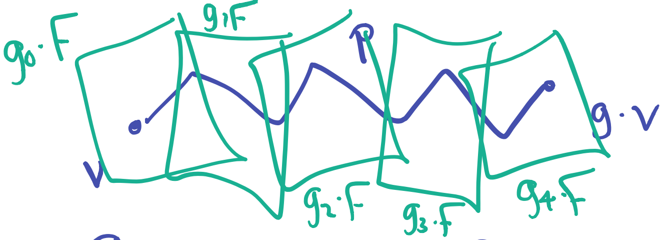
Then S generates G.

Proof. Let $g \in G$

Choose a vertex v in F.

Find a path p from v to $g\cdot v$

(Γ connected)



Choose $g_0, \dots, g_n \in G$

s.t. $g_0 = e$, $g_n = g$

$\bigcup_{i=1}^{n} g_i \cdot F$ contains p.

& $g_i \cdot F \cap g_{i+1} \cdot F \neq \emptyset$ $\forall i$.

Now: $g_0 F \cap g_1 \cdot F \neq \emptyset \implies g_1 \in S$

$g_1 F \cap g_2 F \neq \emptyset$

$\implies F \cap g_1^{-1} g_2 \cdot F \neq \emptyset \implies g_1^{-1} g_2 \in S$

$\implies g_2$ is a product of two elts of S.....