ANNOUNCEMENTS FEB 4

- Cameras on
- Grade/topic due Fri Gradescope
- HW 3 due Feb 11 3:30
- Abstracts due Feb 26
- Office hours Fri 2-3, Tue 11-12, appt.

Today: Generators from group actions. $SL_2\mathbb{Z}$

Fundamental domains ~ existence, . . .

Thm $G \curvearrowright \Gamma$ connected

$F \subseteq \Gamma'$ subgraph.

$$\bigcup_{g \in G} g \cdot F = \Gamma'$$

Then $S = \{ g \in G : g \cdot F \cap F \neq \emptyset \}$

generates G.

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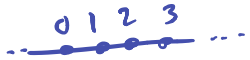
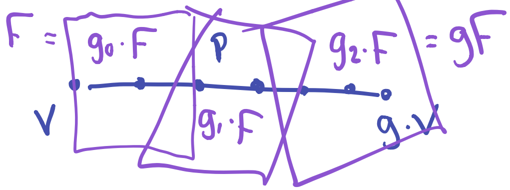
example: $\mathbb{Z} \curvearrowright$ 

$F =$  $S = \{ \pm 1, 0 \}$

Pf. Let $g \in G$. Pick v = vertex of F.

Choose a path p from v to $g \cdot v$

(Γ connected)



Choose $g_0 \cdot F, \dots, g_n \cdot F$

s.t. $g_0 = e$, $g_n = g$, $p \subseteq \bigcup_{i=1}^{n} g_i \cdot F$

$g_i \cdot F \cap g_{i+1} \cdot F \neq \emptyset$.

Show by induction: g_i is a prod. of elts of $S^{\pm 1}$.

$i = 0$ ✓

Assume true for i. WTS for $i+1$.

$g_{i+1} \cdot F \cap g_i \cdot F \neq \emptyset$

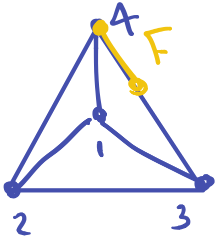
$\Longrightarrow g_i^{-1} \cdot g_{i+1} \cdot F \cap F \neq \emptyset$

$\Longrightarrow g_i^{-1} \cdot g_{i+1} = s \in S \Longrightarrow g_{i+1} = g_i s$ □

Example 1. $S_n \circlearrowright K_n$

$S_n = \mathrm{Sym}_n$

F = half-edge from n to n-1.



S contains: $\mathrm{Stab}(n) = S_{n-1}$

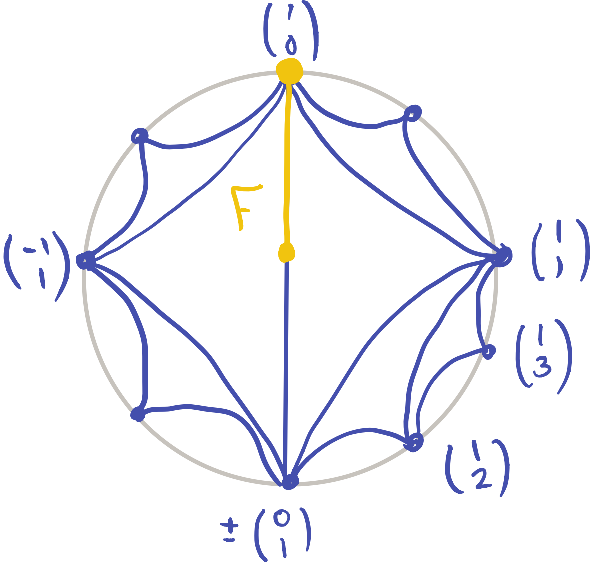
$(n-1\ \ n)$ · ~~any elt of S_{n-2}~~

Can simplify: S_{n-1} ← by induction: gen by adjacent transpositions.

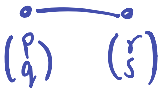
$(n-1\ \ n)$

$\Longrightarrow$ S_n gen by adjacent transp.

Example 2. $SL_2\mathbb{Z} \circlearrowright$ Farey graph.



vertices : $\{$primitive $\mathbb{Z}$ vectors$\}/\pm$

edges :  $\Longleftrightarrow \det\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1.$

$\Longleftrightarrow$ integer bases for $\mathbb{Z}^2$

Note: $\exists\, A \in SL_2\mathbb{Z}$ s.t. $A\cdot\binom{1}{0} = \binom{0}{1}$

$$A = \begin{pmatrix} 0 & \overset{\text{must be } -1}{*} \\ 1 & \underset{\text{anything.}}{*} \end{pmatrix}$$

Better: $\exists\, A \in SL_2\mathbb{Z}$ s.t. $A\cdot\binom{1}{0} = \binom{p}{q}$

$$A = \begin{pmatrix} p & * \\ q & * \end{pmatrix}$$ Bezout

$$\underline{*}\,p + \underline{*}\,q = 1$$

$\Longrightarrow$ F need only 1 vertex of Γ....

Note: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ flips vertical edge.
↳ in S!

Also need: Stab $\binom{1}{0}$.

What about Stab$\begin{pmatrix}1\\0\end{pmatrix}$?

$$\xcancel{\begin{pmatrix}1 & p\\0 & q\end{pmatrix}} \quad \xcancel{\begin{pmatrix}1 & n\\0 & 1\end{pmatrix}}$$

$$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & n\\0 & 1\end{pmatrix}$$

only need: $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$

and: $\xcancel{\begin{pmatrix}-1 & -1\\0 & -1\end{pmatrix}}$ $-I$

(because first col is really $\begin{pmatrix}\pm 1\\0\end{pmatrix}$)

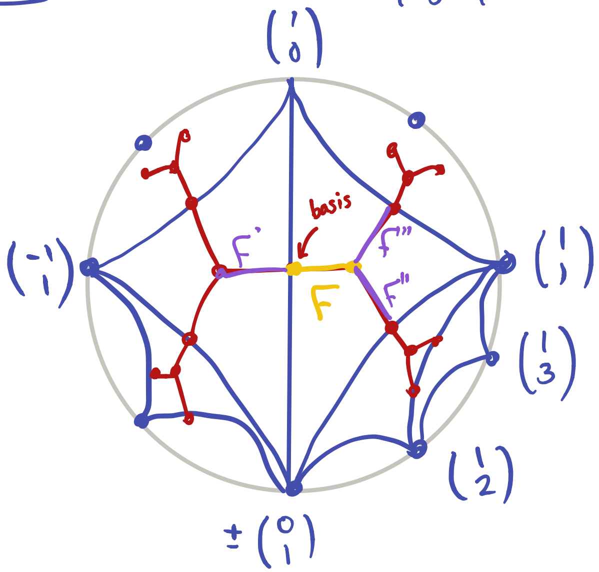
Finally: $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$

$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$

$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$

generates $SL_2\mathbb{Z}$.

Example ~~2~~ 3. $SL_2\mathbb{Z} \circlearrowright$ Farey ~~graph~~ tree.



What is F?

gen set?

$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ takes F to F'

$\begin{pmatrix}0&1\\-1&1\end{pmatrix}$ takes F to F'' & F'''

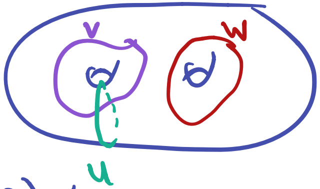
$\cancel{\begin{pmatrix}-1&0\\0&-1\end{pmatrix}}$ fixes F (and the whole tree)

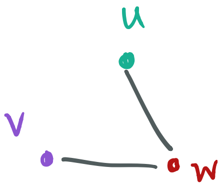
These gens have finite order:

4, 6, ~~2~~.

A more far out example.

Take S = surface





$\mathrm{MCG}(S) = \mathrm{Homeo}(S)/\text{homotopy}.$

We find generators using curve graph

vertices: simple closed curves in S./homotopy

edges: disjointness