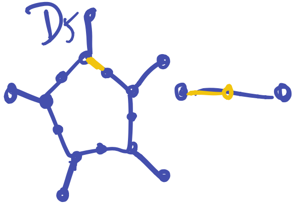
ANNOUNCEMENTS FEB 9

- Cameras on
- Abstract Feb 26 (consult w/ me)
- HW 3 due Thu 3:30
- OH Fri 2-3, appt

Today: Fundamental domains

D_∞.

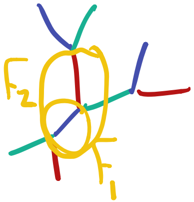
Fundamental domains



Have $G \circlearrowright \Gamma$

F = minimal, connected subgraph of Γ' so

$$\bigcup g \cdot F = \Gamma'.$$



Lemma. If $G \circlearrowright \Gamma$ = connected graph then the action has a fund dom. F.

Pf. First assume $G \circlearrowright \Gamma'$ has finitely many orbits of edges. (*)

Color each edge according to orbit.

Example: $D_5 \circlearrowright$ 10-gon has 1 orbit of edges

$\mathbb{Z}/5 \circlearrowright$ 10-gon has 2 orbits. of edges

Build F inductively.

Choose any edge. Call it F.

Find a new color edge (not in F) adjacent to F, and add it to F.

This stops by (*)

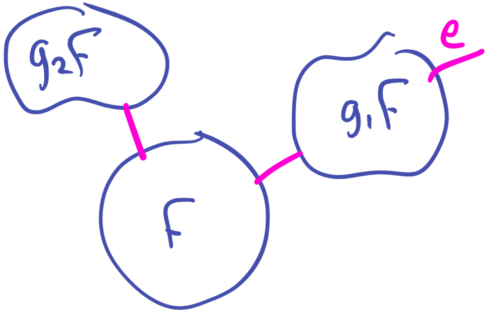
Need to show

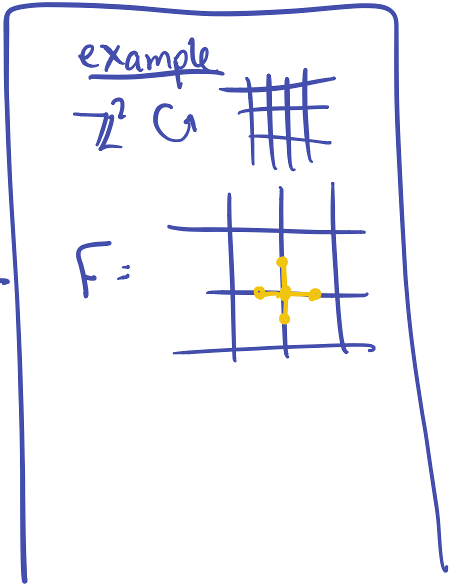
$$\bigcup_{g\in G} g\cdot F = \Gamma'$$

Suppose not. There is an edge e not in $\bigcup g\cdot F$ and adjacent to it.

Say e adjacent to gF.

Then $g^{-1}e$ is adjacent to F.





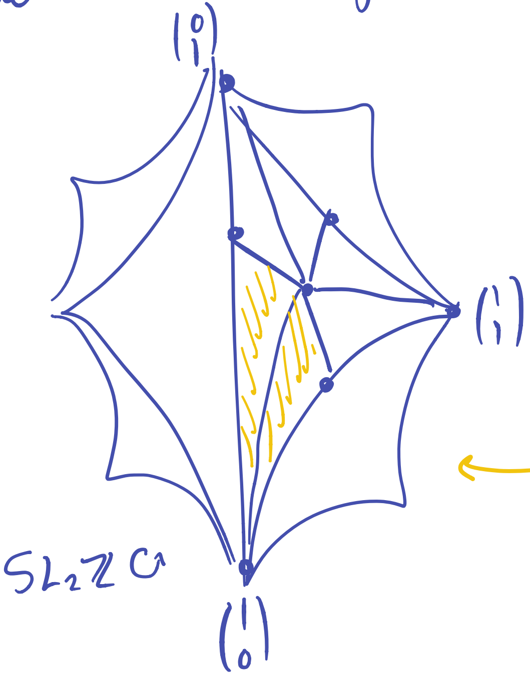
This is a contradiction. We should have added $g^{-1}e$ already □

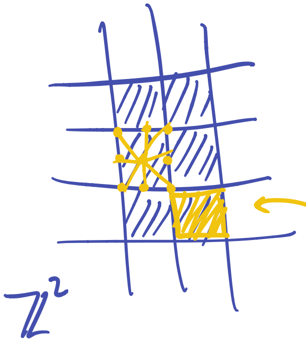
We proved: F is a union of edges from different G-orbits.

So: any two translates of F can only meet at vertices.

So: the $\{g\cdot F\}$ "tile" Γ'.

Aside: There is a higher dimensional version.





2D cell complex.

Fundamental domain

Thm. Say $G \curvearrowright \Gamma$ = conn. graph

$H \leq G$

$F_G \subseteq \Gamma'$ fund. dom for G

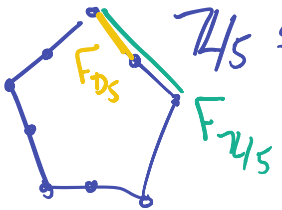
$F_H \subseteq \Gamma'$ fund dom for H

If $F_H = g_1 F_G \cup \cdots \cup g_n F_G$

then $[G:H] = n$.

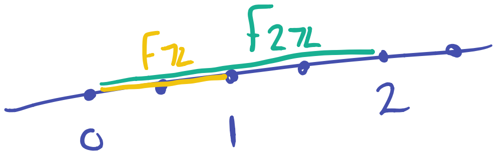
Examples

① $\mathbb{Z}/5 \leq D_5$



② $n\mathbb{Z} \leq \mathbb{Z}$

$n = 2$



Thm. Say $G \curvearrowright \Gamma$ = conn. graph

$H \leq G$

$F_G \subseteq \Gamma'$ fund. dom for G

$F_H \subseteq \Gamma'$ fund dom for H

If $F_H = g_1 F_G \cup \cdots \cup g_n F_G$

then $[G:H] = n$.

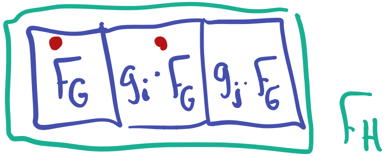
Pf. Define

$$\{g_i \cdot F_G\}_{i=1}^n \longrightarrow \{G/H\}$$

$$g_i \cdot F_G \longmapsto g_i H$$

Want: bijection

A picture:



Injectivity Suppose $g_i H = g_j H$

$\Longrightarrow g_i^{-1} g_j \in H$

$\Longrightarrow g_i^{-1} g_j$ does not identify two pts in interior of F_H

$\Longrightarrow g_i \cdot F_G = g_j \cdot F_G$

otherwise $g_i^{-1} g_j$ takes $g_j^{-1} \cdot F_G$ to $g_i^{-1} \cdot F_G$.

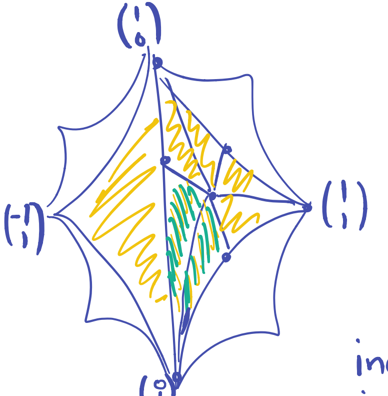
FINISH

An application

$SL_2(\mathbb{Z})[m]$ = level m congruence subgp of $SL_2\mathbb{Z}$

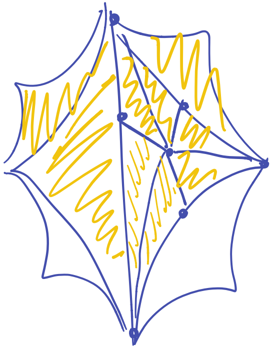
$= \{ A \in SL_2(\mathbb{Z}) : A \equiv I \bmod m \}.$

In $SL_2(\mathbb{Z})[2]$: $\pm I$, $\begin{pmatrix} 3 & 10 \\ 2 & 7 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

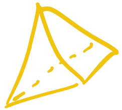


$SL_2(\mathbb{Z})[2]$

index 6 in $SL_2(\mathbb{Z})$



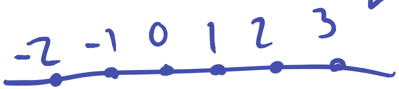
$SL_2(\mathbb{Z})[3]$



index 12 in $SL_2\mathbb{Z}$

Chap 2. Groups gen. by reflections

(Coxeter groups)

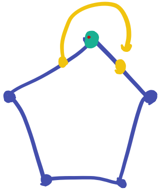
Infinite Dihedral Gp = Sym() = D_∞

∞-gon.

Sample elts: translate by n

reflect in vertex

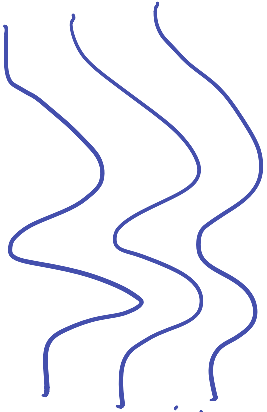
reflect about middle of edge.



It is gen by reflections about 0, $1/2$.

More next time!

$t^2 = id.$ ~~$tat^{-1} = ?$~~



$$t_i := (i\ \ i{+}1) = a^i t a^{-i}$$

$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$$

$$a^i t a^{-i} a^{i+1} t a^{-i-1} a^i t a^{-i} = a^{i+1} t a^{-i-1} a^i t a^{-i} a^{i+1} t a^{-i-1}.$$