Announcements Feb 9

- Cameras on
- Abstract Feb 26 (consult w/me)
- HW 3 due Thu 3:30
- OH Fri 2-3, appt

Today: Fundamental domains
D∞.
Fundamental domains

Have $G \leq \Gamma$

$F = \text{minimal, connected subgraph of } \Gamma'$ so $Ug \cdot F = \Gamma'$

Lemma. If $G \leq \Gamma$ connected graph then the action has a fund dom. $F$.

Proof. First assume $G \leq \Gamma'$ has finitely many orbits of edges.

Color each edge according to orbit.

Example: $D_5 \leq 10$-gon has 1 orbit of edges

$Z/5 \leq 10$-gon has 2 orbits of edges

Build $F$ inductively.

Choose any edge. Call it $F$.

Find a new color edge (not in $F$) adjacent to $F$, and add it to $F$.

This stops by $\ast$
Need to show $Ug \cdot F = \Gamma'$

$e \in G$

Suppose not. There is an edge $e$ not in $Ug \cdot F$ and adjacent to it.

Say $e$ adjacent to $gF$

Then $g^{-1}e$ is adjacent to $F$.

This is a contradiction.

We should have added $g^{-1}e$ already

We proved: $F$ is a union of edges from different orbits.

So: any two translates of $F$ can only meet at vertices.

So: the $\{g \cdot F\}$ "tile" $\Gamma'$. 

\[\text{example} \quad \begin{array}{|c|c|c|} \hline \quad & \quad & \quad \\ \hline \quad & \quad & \quad \\ \hline \end{array} \]

\[F = \begin{array}{|c|c|} \hline \quad & \quad \\ \hline \quad & \quad \\ \hline \end{array} \]
Aside: There is a higher dimensional version.

2D cell complex.

Fundamental domain
Thm. Say $G \vartriangleleft \Gamma = \text{conn. graph}$

$\Delta \leq G$

$\Gamma' \leq \Gamma'$ fund. dom for $G$

$\Gamma' \leq \Gamma'$ fund. dom for $H$

If $\Gamma' = \varnothing$, $\Gamma \cup \ldots \cup \varnothing = \Gamma$

then $[G : H] = n$.

Examples

1. $\mathbb{Z}^5 \leq D_5$

   \[ F_{D_5} \]

2. $n\mathbb{Z} \leq \mathbb{Z}$

   \[ F_{n\mathbb{Z}} \]  
   \[ F_{2\mathbb{Z}} \]  
   \[ 0 \quad 1 \quad 2 \quad n=2 \]
Thm. Say $G \subset \Gamma = \text{conn. graph}$
$H \leq G$
$F_G \subset \Gamma'$ fund. dom for $G$
$F_H \subset \Gamma''$ fund dom for $H$
If $F_H = g_i F_G \cup \cdots \cup g_n F_G$
then $\left[ G : H \right] = n$.

Pf. Define
$\{g_i : F_G\}_{i=1}^n \rightarrow \{G/H\}$
$g_i : F_G \mapsto g_i H$

Want: bijection

A picture: $\begin{array}{cccccc}
& & & F_G & g_i : F_G & g_j : F_G \\
& & & & & F_H \\
\end{array}$

Injectivity: Suppose $g_i H = g_j H$
$\Rightarrow g_i^{-1} g_j \in H$
$\Rightarrow g_i^{-1} g_j$ does not identify two pts in int of $F_H$
$\Rightarrow g_i : F_G = g_j : F_G$
otherwise $g_i^{-1} g_j$ takes $g_j : F_G$
to $g_i^{-1} : F_G$
$\boxed{\text{FINISH}}$
An application

\[ SL_2(\mathbb{Z})[m] = \text{level } m \text{ congruence subgp of } SL_2(\mathbb{Z}) = \left\{ A \in SL_2(\mathbb{Z}) : A \equiv I \mod m \right\}. \]

In \( SL_2(\mathbb{Z})[2] \):

\[ \pm I, \ \begin{pmatrix} 3 & 1 \\ 2 & 7 \end{pmatrix}, \ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \]

\[ SL_2(\mathbb{Z})[2] \] in \( SL_2(\mathbb{Z}) \) has index 6.

\[ SL_2(\mathbb{Z})[3] \] in \( SL_2(\mathbb{Z}) \) has index 12.
Chap 2. Groups gen. by reflections  
(Coxeter groups)

Infinite Dihedral Gp = \text{Sym} \left( \begin{array}{cccc}
-2 & -1 & 0 & 1 \\
\end{array} \right) = D_{\infty}

Sample elts: translate by \( n \)
reflect in vertex
reflect about middle of edge.

It is gen by reflectors about \( 0, \frac{1}{2} \).

More next time!
\( t^2 = \text{id.} \)

\( \text{tat} = ? \)

\[ t_i = (i \: i+1) = a^i t a^{-i} \]

\[ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1} \]

\[ a^i t a^{-i} a^{i+1} t a^{-i} a t a^{-i} = a^{i+1} t a^{-i} a^i t a^{-i} a^{i+1} \]