Announcements Feb 11

- Cameras on
- Abstracts Feb 26: consult with me
- HW4 due Thu 3:30
- Office hours Fri 2-3, Tue 11-12, appt

Today: Do
- Triangle gps
- Coxeter groups.
From last time:

**Thm:** If $G 	riangleleft \Gamma$

- $\text{fund dom } F.$ and $g \cdot F = f$
- $H \leq G \implies g = \text{id.}$
- $\text{fund dom } F_H$
- $F_H = g_1 F \cup \ldots \cup g_n F$

Then $[G : H] = n.$

*Index* 2

e.g. $\mathbb{Z} \leq \mathbb{Z} \triangleleft \mathbb{R}$

$\mathbb{Z} \times 1 \leq \mathbb{Z} \times \mathbb{Z}/2$ *Index* 4

Noah's question:

Take $G \triangleleft \Gamma.$

Index $n$ $H \leq G$ \quad F

Now: $K$ other gp.

$G \times K \triangleleft \Gamma$

$H \times 1 \leq G \times K$ index bigger.

Same fund domains as before?

If yes: seems like contradiction.

$\text{Fix}$
Infinite Dihedral Group

\( \Gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \)

\( \text{Doo} = \text{Sym}(\Gamma) \)

Last time: gen. by

\( a = \text{refl. about } 0 \)
\( b = \text{refl. about } \frac{1}{2} \).

Presentation?

To start:

\( a^2 = b^2 = \text{id} \)

What else?

Typical elt of gp:

\( abab^{-1} b^{-4} bab' \)

really, this is:

\( abababab \to a \)

alternating word in \( a, b \).

So all elts are:

\( (ab)^n \quad (ab)^n a \quad (ba)^n \quad (ba)^n b \quad n > 0 \).

Presentation: \( \text{Doo} \cong \langle a, b \mid a^2 = b^2 = \text{id} \rangle \)
A subgp of $D_\infty$

$H = \langle a, bab \rangle = \text{subgp gen by } a, \text{ bab. in } D_\infty.$

$H$ is isomorphic to $D_\infty$

$[D_\infty : H] = 2.$

By the way:

$H = \text{kernel if }$ $D_\infty \to \mathbb{Z}/2$

"count # of b's mod 2"

An explicit $H \to D_\infty$

$a \mapsto a$

$bab \mapsto b$
Triangle groups

$W_{333} = \text{gp gen. by reflections in } r, b, g, \text{ and } gb$.

What are $g$, $grg$, $gbg$?  
- Mult. choice.
- $rb$?
- No choice.
  
grg = reflection about image of $r$ under $g$.

$gbg$

$rb = \text{rotation by } 2\pi/3$

Some relations: $r^2 = b^2 = g^2 = \text{id}$,
\[(rb)^3 = (rg)^3 = (gb)^3 = \text{id}.

Goals: Fund. domain and presentation.
Guess for fund domain: original triangle.

To this end... take tiling of $E^2$ by $\triangle$

& color the edges:

Critical point: this coloring is well-defined.
Each color is tiling by regular hexagons. Check: g,b, r,g,b present three 3 hexagonal tilings.
We just showed

Prop. The coloring is well defined.

Cor. If \( g \in W_{333} \) & \( g \cdot T_0 = T_0 \)
then \( g = \text{id} \).

So the fund domain is at least as big as \( T_0 \).

To show \( T_0 \) is a fund domain,
need that \( W_{333} \) acts trans. on triangles. Equivalently \( W_{333} \cdot T_0 = H^2 \).

Prop. Let \( T \) be a triangle of the tessellation.
and \( T_0, T_1, \ldots, T_n = T \)
is a seq of triangles s.t. \( T_i \cap T_{i+1} \) is an edge
colored \( C_i \in \{r,g,b\} \).

Then \( C_1 \cdots C_n \cdot T_0 = T \).
Prop. Let $T$ be a triangle of the tesselation. 

and $T_0, T_1, \ldots, T_n = T$ is a seq of triangles s.t. $T_i \cap T_{i+1}$ is an edge colored $c_i \in \{r, g, b\}$. Then $c_1 \ldots c_n \cdot To = T$.

Pf. Induct on $n$.

$n = 0 \checkmark$

Inductive hyp: $c_1 \ldots c_{n-1} \cdot To = T_{n-1}$

Define $T' :$ 

Note $T' = c_n To$

Have $c_1 \ldots c_{n-1} T' = T_n$

$c_1 \ldots c_{n-1} c_n To = T_n \square$
Coxeter groups: all generators have order 2.
all other relations:

\[(ab)^n = \text{id}\]

e.g. \(D_n\).