Announcements Feb 11

- Cameras on
- Abstracts Feb 26 : consult with me
- HW4 due Thu 3:30
- Office hours Fri 2-3, Tue 11-12, appt

Today: D_∞

Triangle gps

Coxeter groups.

From last time:

Thm. If $G \curvearrowright \Gamma$

fund dom F.

and $g \cdot F = F$
$\Rightarrow g = \mathrm{id}.$

$H \leq G$

fund dom F_H

and $F_H = g_1 \cdot F \cup \dots \cup g_n \cdot F$

then $[G : H] = n$.

index 2

e.g. $2\mathbb{Z} \leq \mathbb{Z} \curvearrowright$ 

$2\mathbb{Z} \times 1 \leq \mathbb{Z} \times \mathbb{Z}/2$ index 4

Noah's question:

Take $G \curvearrowright \Gamma$. $\qquad F$

index n $H \leq G$ $\qquad F_H$

Now: K other gp.

$G \times K \curvearrowright \Gamma$

$H \times 1 \leq G \times K$ index bigger.

Same fund domains as before?

If yes: seems like contradiction.

Fix

Infinite Dihedral Group

$\Gamma =$

$D_\infty = \mathrm{Sym}(\Gamma)$

Last time: gen. by

$a =$ refl. about 0

$b =$ refl. about $1/2$.

Presentation?

To start: $a^2 = b^2 = id$

What else?

Typical elt of gp:

~~$aba^{-1}b^7a^4bab^{-1}$~~

~~really, this is:~~

~~$ababebab \rightsquigarrow a$~~

alternating word in a, b.

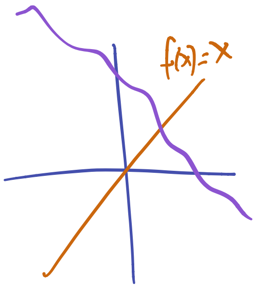
So all elts are:

translation by n ← $(ab)^n$ $\quad$ $(ab)^n a$ ← reflections by IVT

translation by $-n$. ← $(ba)^n$ $\quad$ $(ba)^n b$ ← reflections by IVT

about what?

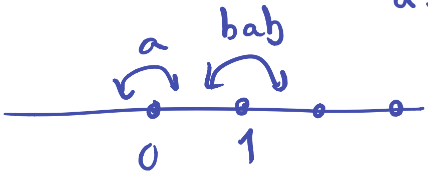
$n \geq 0$.



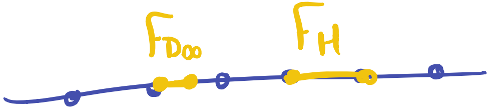
Presentation: $D_\infty \cong \langle a, b \mid a^2 = b^2 = id \rangle$

A subgp of D_∞

$H = \langle a, bab \rangle$ = subgp gen by a, bab. in D_∞.



H is isomorphic to D_∞



$[D_\infty : H] = 2$.

By the way:

H = kernel of

$$D_\infty \to \mathbb{Z}/2$$

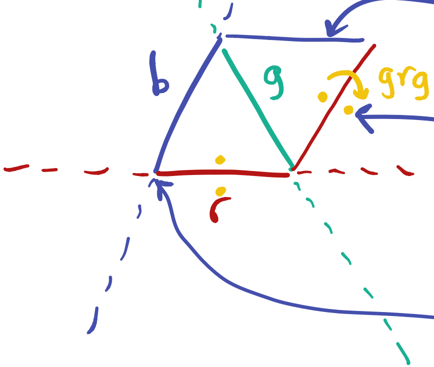
"count # of b's mod 2"

An explicit $H \to D_\infty$

$$a \longmapsto a$$
$$bab \longmapsto b$$

Triangle groups

W_{333} = gp gen. by reflections in



Goals: Fund. domain.
Presentation

What are g, grg, gbg? mult. choice.

rb? no choice.

grg = reflection about image of r under g.

gbg

rb = rotation by $2\pi/3$

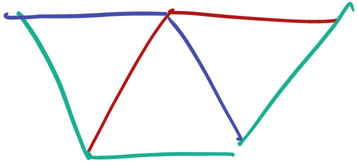
Some relations: $r^2 = b^2 = g^2 = id$

$(rb)^3 = (rg)^3 = (gb)^3 = id.$

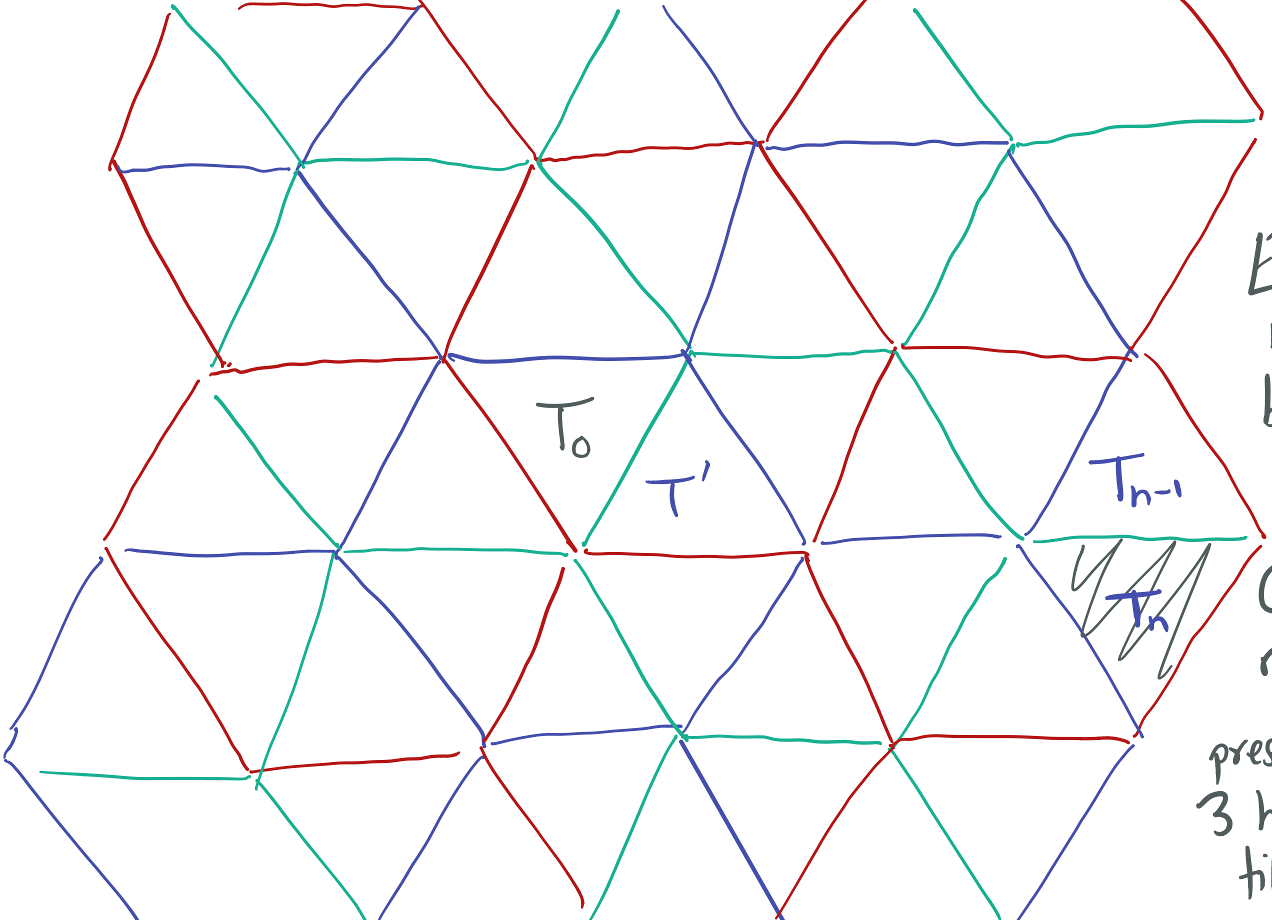
Guess for fund domain: original triangle.

To this end... take tiling of $\mathbb{E}^2$ by $\triangle$

& color the edges:



Critical point: this coloring is well-defined.



Each color is tiling by regular hexagons.

Check: r, g, b preserve these 3 hexagonal tilings.

We just showed

Prop. The coloring is well defined.

Cor. If $g \in W_{333}$ & $g \cdot T_0 = T_0$ then $g = \mathrm{id}$.

So the fund domain is at least as big as T_0.

To show T_0 is a fund domain, need that W_{333} acts trans. on triangles. Equivalently $W_{333} \cdot T_0 = \mathbb{H}^2$.

Prop. Let T be a triangle of the tesselation, and $T_0, T_1, \dots, T_n = T$ is a seq of triangles s.t. $T_i \cap T_{i+1}$ is an edge colored $c_i \in \{r, g, b\}$. Then $c_1 \cdots c_n \cdot T_0 = T$.

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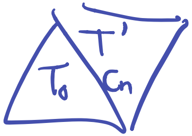
Pf. Induct on n.

$n = 0$ ✓

Inductive hyp:

$$c_1 \cdots c_{n-1} \cdot T_0 = T_{n-1}$$

Define T' :



Note $T' = c_n T_0$

Have $c_1 \cdots c_{n-1} T' = T_n$

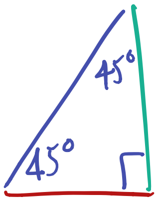
$$c_1 \cdots c_{n-1} c_n T_0 = T_n \qquad \square$$

Coxeter groups: all generators have order 2.

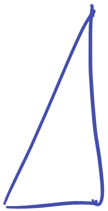
all other relations:

$$(ab)^n = \mathrm{id}.$$

e.g. D_n.



W_{244}



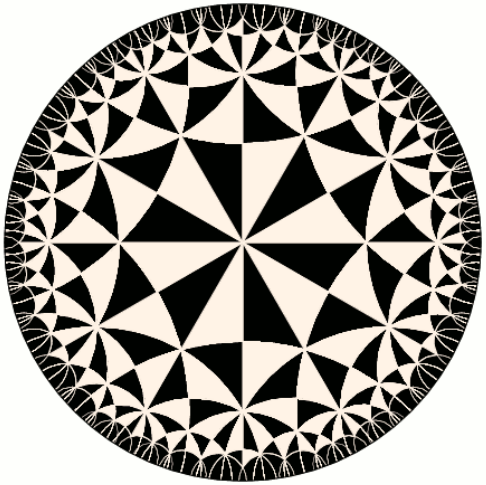


Figure 7 from Coxeter's address to the Royal Society of Canada