ANNOUNCEMENTS FEB 19

- Cameras on.
- HW 4 due Thu 3:30
- Abstracts Feb 26 : consult with me before Feb 26.
- Take home midterm Mar 4
- Fri office hours moved (requests?)

  Office hours Tue 11-12, appt.

Sample HW Solution

21. Let H be the subset of $\mathrm{Sym}_{\mathbb{Z}}$ so that $h \in H$ iff $\exists$ finite $C \subset \mathbb{Z}$, $k \in \mathbb{Z}$ s.t. $h(n) = n+k$ for $n \notin C$.

state problem.

(a) Show that H is a subgroup of $\mathrm{Sym}_{\mathbb{Z}}$.

white space!

identity: We see $e \in \mathrm{Sym}_{\mathbb{Z}}$ belongs to H by taking $C = \emptyset$, $k = 0$.

Scannable!

systematic!

inverses: Let $h \in H$ with associated C, k. Let $C' = C + k$, $k' = -k$. Then h^{-1}, C', k' satisfy the required

headings!

conditions. Indeed: $|C'| = |C| < \infty$ and

$$
\begin{aligned}
& n \notin C' \\
\Rightarrow\ & n-k \notin C \\
\Rightarrow\ & h(n-k) = n \\
\Rightarrow\ & h^{-1}(n) = n-k = n+k'.
\end{aligned}
$$

sequence of implications! (aligned)

details!

<u>Composition</u>. Let $h_1, h_2 \in H$ with associated C_1, k_1 & C_2, k_2.

Let $C' = (C_1 - k_2) \cup C_2$, $k' = k_1 + k_2$.

Then $h_1 h_2, C', k'$ satisfy the required conditions since

parallel structure!

$$|C'| = |(C_1 - k_2) \cup C_2| \leq |C_1 - k_2| + |C_2| = |C_1| + |C_2| < \infty$$

and $\quad n \notin C'$

$$\Rightarrow n \notin C_2 \text{ and } n \notin C_1 - k_2$$

$$\Rightarrow n \notin C_2 \text{ and } n + k_2 \notin C_1$$

$$\Rightarrow h_1 h_2(n) = h_1(n + k_2) = n + k_1 + k_2 = n + k'$$

(b) Show that H is finitely generated.

Let $s = (0 \ \ 1)$

$t = (\cdots -1 \ 0 \ 1 \ 2 \cdots)$

We will show that s, t generate H.

<u>Claim</u> $(i \;\; i{+}1) = t^i s t^{i-1}$

Claim!

<u>Pf of Claim</u>.

Editing!

Let $h \in H$ with associated c, k.

Note $t^{-k}h$ has associated $c' = c - k$, $k' = 0$.

So $t^{-k}h$ can be regarded as an element of

$\mathrm{Sym}_{c'} \subseteq \mathrm{Sym}_{\mathbb{Z}} \ldots$

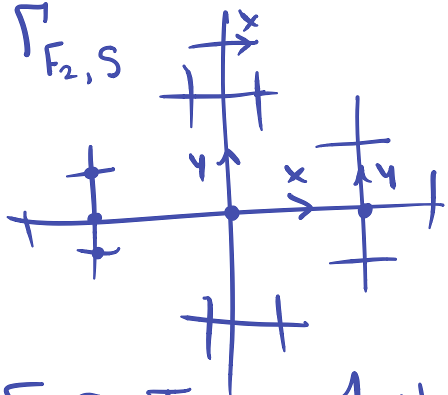
Chap 3 Groups acting on trees.

3.1 Free groups.

$F_2 = \langle x, y \mid \, \rangle$

$S = \{x, y\}$

$\Gamma_{F_2, S}$



So $F_2 \curvearrowright T_4$ = reg. 4 valent tree.

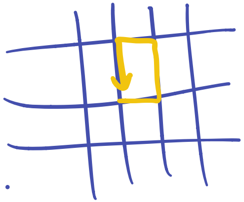
Why?

Non-backtracking Paths in Cayley graph.

$\longleftrightarrow$ reduced words in x, y.

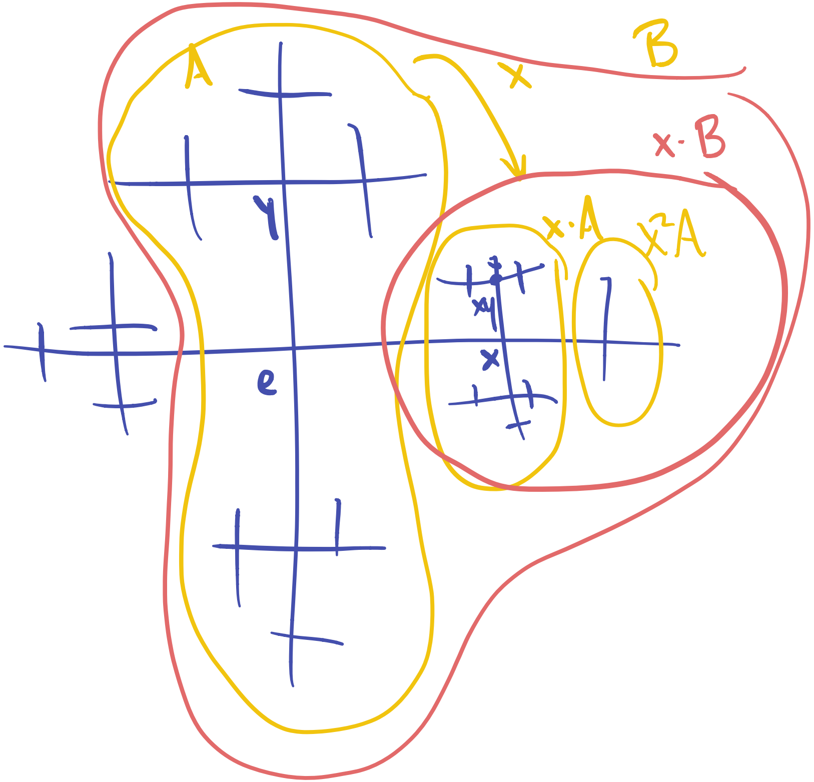
For free group with free gen set: no loops in Cayley graph.

Or: relations among generators $\longleftrightarrow$ circuits in Cayley graph.

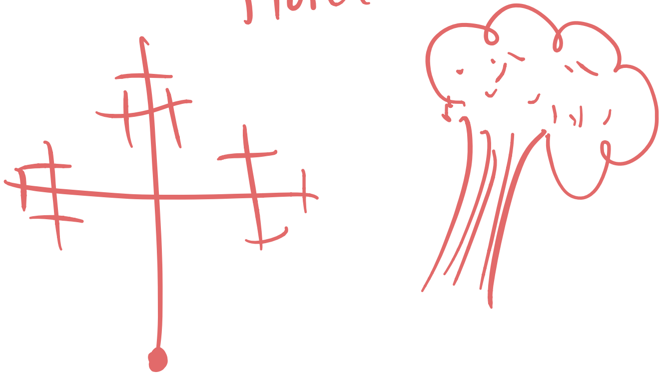


The action $F_2 \circlearrowright T_4$

What does x do?



Floret



Goal Let $x = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Then x, y generate a subgp of $SL_2\mathbb{Z}$, denoted $\langle x, y \rangle$

Thm. $\langle x, y \rangle \cong F_2$.

In other words, every nonempty freely reduced word in $x^{\pm 1}, y^{\pm 1}$ multiplies to a nontrivial matrix.

False if you replace the 2's with 1's.

Indeed...

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

$$= SL_2\mathbb{Z}.$$

proof: row reduction.

which is not free

because... torsion

Exercise. Free groups are torsion free.

Ping Pong Lemma

Say $G \curvearrowright X$ = set

$a, b \in G$

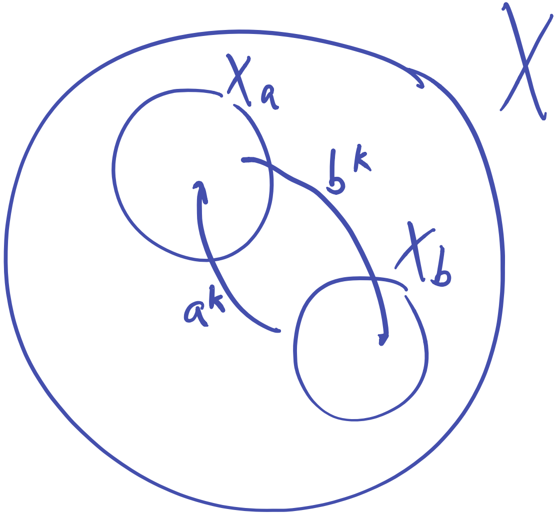
$X_a, X_b \subseteq X$

nonempty, disjoint

$a^k \cdot X_b \subseteq X_a \quad \forall k \neq 0$

$b^k \cdot X_a \subseteq X_b \quad \forall k \neq 0$

Then $\langle a, b \rangle \cong F_2$.



Pf by example

Q. Why is $abab^2a^3 \neq id$?

A. For any $x \in X_b$

$abab^2a^3 \cdot x$ in X_a hence $\neq x$.

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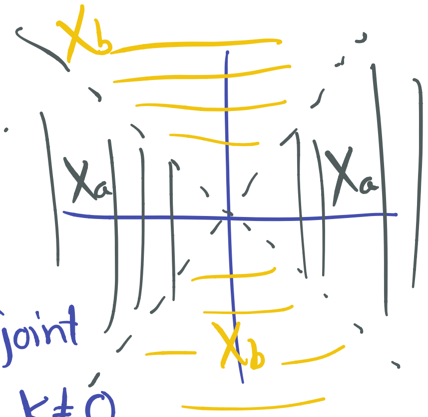


Application 1

$G = SL_2(\mathbb{Z})$

$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$X = \mathbb{Z}^2$

$X_a = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2 : |p| > |q| \right\}$

$X_b = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{Z}^2 : |q| > |p| \right\}$

Check: If $\begin{pmatrix} p \\ q \end{pmatrix} \in X_b$, $k \neq 0$ then

$a^k \cdot \begin{pmatrix} p \\ q \end{pmatrix} \in X_a$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -99 \\ 101 \end{pmatrix} = \begin{pmatrix} 103 \\ 101 \end{pmatrix}$$

Check: If $\binom{p}{q} \in X_b$, $k \neq 0$ then

$$a^k \cdot \binom{p}{q} \in X_a$$

$$a^k \binom{p}{q} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k \binom{p}{q}$$

$$= \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \binom{p}{q}$$

$$= \begin{pmatrix} p + 2kq \\ q \end{pmatrix}$$

But $|p + 2kq| \geq |2kq| - |p|$

$$= 2|k||q| - |p|$$

$$> 2|k||q| - |q|$$

$$> |q|$$

$\square$.

Application 2. Homeo($\mathbb{R}$)

Homeo($\mathbb{R}$) = $\{ f : \mathbb{R} \to \mathbb{R}$:

f contin with

(contin) inverse $\}$

group op: $f \circ g$

Poll. If $f : \mathbb{R} \to \mathbb{R}$ continuous & bij

is f^{-1} automatically contin.

Yes

Invertible: horiz. & vert.

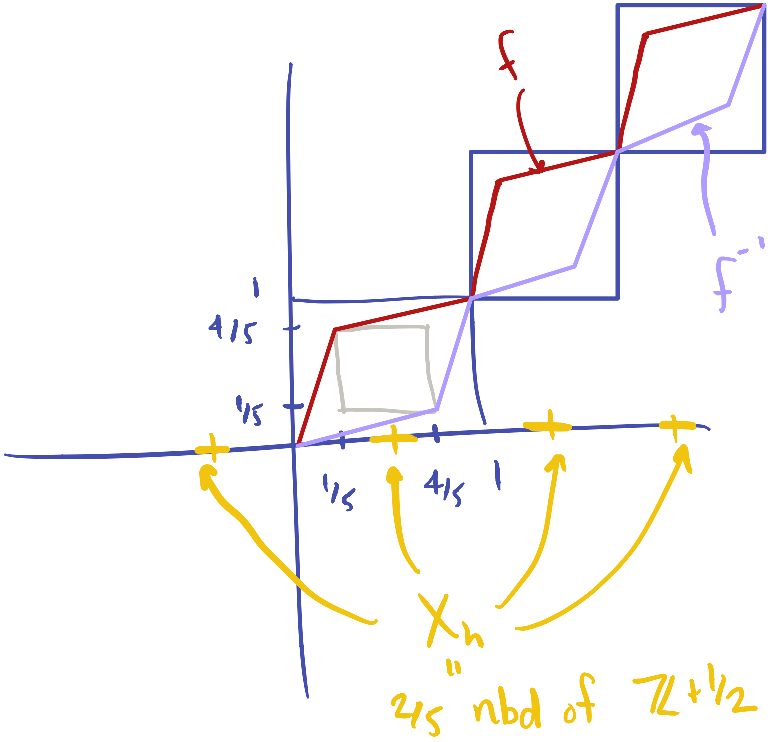
line test.

Inverse: flip over $y = x$.

Goal: $F_2 \leq$ Homeo($\mathbb{R}$).

Let $f(x)$ be:

$X_f = 2/5$ nbd of $\mathbb{Z}$



$g(x)$ is same,

shifted right by $1/2$

up by $1/2$

$$g(x) = f(x - 1/2) + 1/2$$

Prop. $\langle f(x), g(x) \rangle \cong F_2$

Ping pong! $X = \mathbb{R}$

$$X_f = \bigcup_{n \in \mathbb{Z}} [n - 1/5, n + 1/5]$$

$$X_h = \bigcup_{n \in \mathbb{Z}} [n - 1/5 + 1/2, n + 1/5 + 1/2]$$

Next time: $F_3 \leq F_2$ (and $F_2 \leq F_3$)

index 2. ∞ index

& $F_\infty \leq F_2$

$\mathbb{Z} \leq \mathbb{Z}$

index n.