ANNOUNCEMENTS FEB 18

- Cameras on
- HW 5 due Thu
- Abstracts Feb 26 → consult with me ahead of time by meeting/chat/email
- Midterm Mar 4
- Fri office hour @ 10 (just tomorrow)
- Office hours Tue 11-12, appt.

Today: $F_3 \leq F_2$, $G \curvearrowright T$ freely $\Rightarrow$ G free

Ping pong lemma

$G \curvearrowright X$ = set

$a, b \in G$

$X_a, X_b \subseteq X$ disjoint, nonempty

$b^k(X_a) \subseteq X_b \quad \forall k \neq 0$

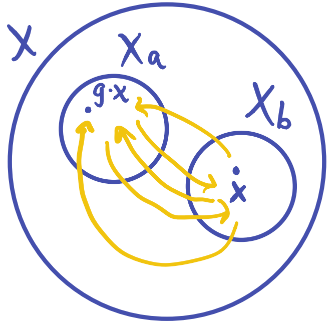
$a^k(X_b) \subseteq X_a \quad \forall k \neq 0$

Then $\langle a, b \rangle \cong F_2$.

Similar: $\langle a_1, \ldots, a_k \rangle \cong F_k$

Pf. If $g = a^* b^* a^* b^* a^*$ freely reduced word in a, b

then $g \neq id$



$g \cdot x \neq x$

$\Rightarrow g \neq id$.

Similar if g starts, ends in b.

If g starts with a, ends with b,

eg. $a^3 b^5 = g$

conjugate so starts, ends with a

$a a^3 b^5 a^{-1} \neq id \Rightarrow g \neq id$. $\square$

3.2 $F_3 \le F_2$

$F_2 = \langle x, y \mid \ \rangle$

H = subset of F_2 consisting of reduced words of even length.

Let $a = x^2$, $b = xy$, $c = xy^{-1}$

Thm. ① $H \trianglelefteq F_2$ of index 2

② H is gen by a, b, c.

③ $H \cong F_3$

Pf. ① Consider

$$F_2 \longrightarrow \mathbb{Z}/2$$

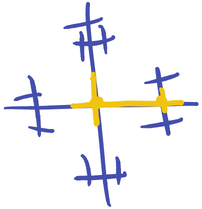
$$g \longmapsto \text{mod } 2 \text{ word length}$$

(or: $x \longmapsto 1$, $y \longmapsto 1$ this defines a homom)

② Write all words of length 2 in $\{x, y\}^{\pm 1}$ in terms of a, b, c.

e.g. $y^2 = c^{-1}b$ etc.

or use our thm... 1 edge + 6 half-edges



Let $a = x^2$, $b = xy$, $c = xy^{-1}$

Thm. ① $H \trianglelefteq F_2$ of index 2

② H is gen by a, b, c.

③ $H \cong F_3$

Pf. ③ Let $w = w_1 \cdots w_n$ freely red. word in a,b,c. Want: $w \neq id$.

Set $w_i = \alpha_i \beta_i$ $\alpha_i, \beta_i \in \{x,y\}^{\pm 1}$

We'll show: If w has a cancellation:

$$\cdots \beta_{i-1} \boxed{\alpha_i} \cancel{\beta_i} \cancel{\alpha_{i+1}} \boxed{\beta_{i+1}} \alpha_{i+2} \cdots$$

then no nearby cancellations:

$$\alpha_i \neq \beta_{i+1}^{-1}$$

$$\beta_{i+1} \neq \alpha_{i+2}^{-1}$$

$$\beta_{i-1} \neq \alpha_i^{-1}$$

In other words: α_i, β_{i+1} don't cancel.

Case by case check.

e.g.

	w_{i-1}	w_i	w_{i+1}
	?	a^{-1}	b
	(5 choices everything except a) ✓	$(x^{-1}\cancel{x^{-1}})$	$(\cancel{x}y)$ (5 choices)
		✓	

□

3.4 Free groups and actions on trees.

Say $G \curvearrowright \Gamma$ = graph.

The action is *free* if

$$g \cdot v = v \Rightarrow g = id.$$

$$\& \quad g \cdot e = e \Rightarrow g = id$$

$$\forall\, g \in G,\ v \in V(\Gamma),\ e \in E(\Gamma)$$

example. $F_2 \curvearrowright T_4$ free.

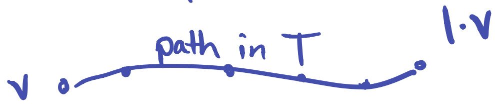
$\mathbb{Z}/n \curvearrowright$ n-cycle free

$D_n \curvearrowright$ n-cycle not free

$G \curvearrowright \Gamma_{G,S}$ free.

Lemma. Any action $\mathbb{Z}/2$ on a tree T is not free.

Pf. v = any vertex



Paths unique $\Rightarrow$ $\mathbb{Z}/2$ preserves the path.

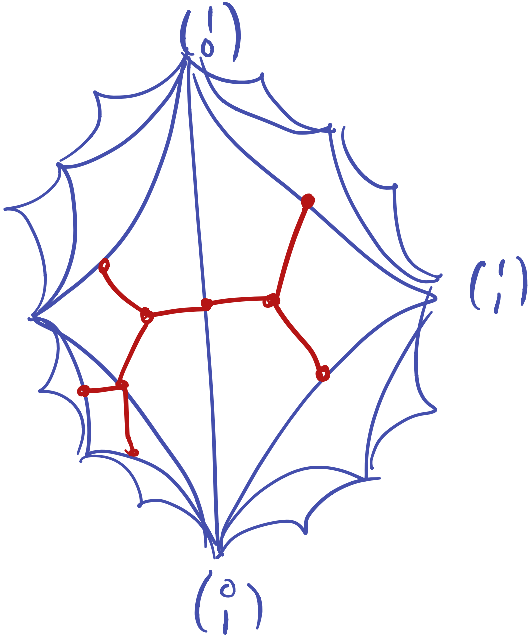
$\Rightarrow$ $\mathbb{Z}/2$ fixes the midpt of path

$\Rightarrow$ fixed edge or vertex. $\square$

Exercise: generalize to $\mathbb{Z}/m$.

Cor. If G has torsion (elt of finite order) then any $G \curvearrowright T$ not free.

<u>Poll</u>: Is $SL_2(\mathbb{Z})[2] \circlearrowright$ Farey tree free?



actually, $-I$ fixes the whole tree.

No. $-I$ has order 2...

Also can find a matrix that "rotates" any vertex

If an elt of $SL_2(\mathbb{Z})$ fixes an edge, it fixes both vertices:

$\mathrm{Stab}(e) \subseteq \mathrm{Stab}(v)$

Thm. If a group acts freely on a tree, it is free.

Cor. Subgroups of free gps a free (hard to prove directly).

Pf #1 of Thm Ping Pong

$G \curvearrowright T$ = tree freely.

F = fundamental dom.

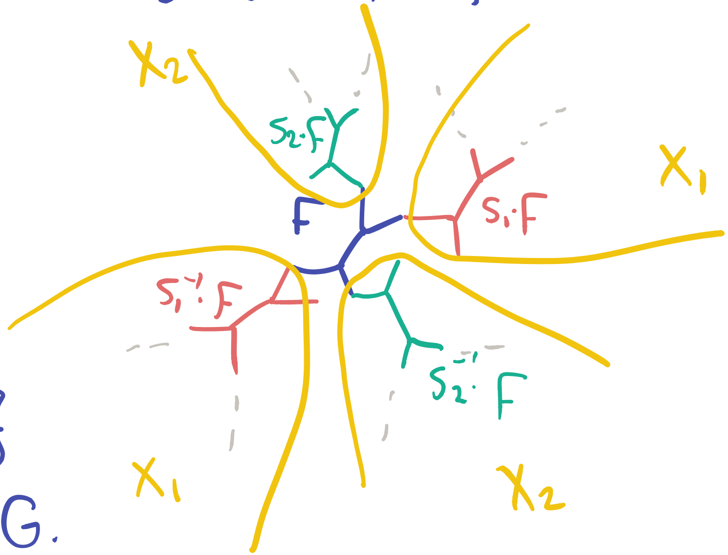
$S = \{g \in G : g \cdot F \cap F \neq \emptyset\}$

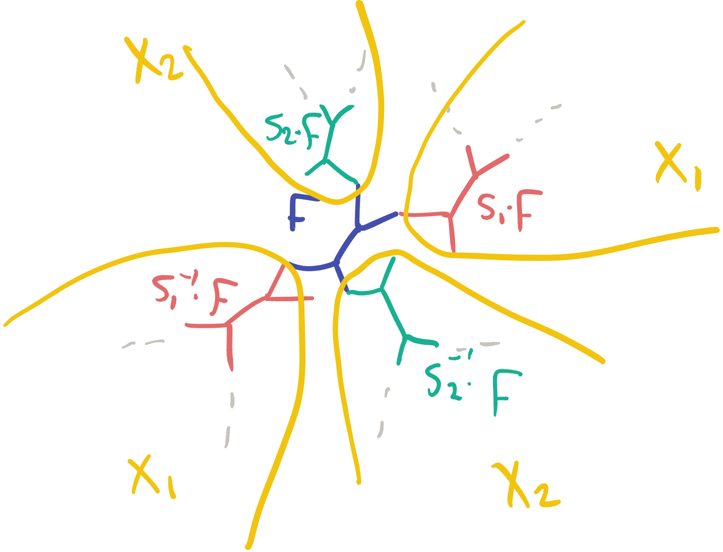
Earlier theorem: S generates G.

From S remove duplicates: x, x^{-1}.

To show: group gen. by S (i.e. G) is free.

$S = \{s_1, s_2, \ldots\}$





To check: $S_2 \cdot X_1 \subseteq X_2$.

$$S_2^{-1}F \xrightarrow[\text{to}]{\text{connected}} F \longrightarrow S_1 F \longrightarrow \overset{\text{rest of}}{X_1}$$

apply S_2:

$$F \longrightarrow S_2 F \longrightarrow (S_2 S_1 F) \longrightarrow S_2 \cdot X_1$$