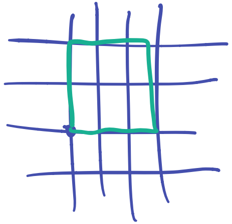
Announcements Feb 23



- Cameras on
- HW5 due Thu
- Abstracts Fri Feb 26 Gradescope (Team submissions)
- Take home midterm March 4
- Office Hours moved to 1:00 Thu
- Regular office hours Tue 11, appt
- Ask for help on HW!

Today

- Ping pong lemma
- Free actions on trees $\leftrightarrow$ free groups
- Free actions on edges of trees $\leftrightarrow$ free products

Ping Pong Lemma II

Lemma 3.10

Have $G \circlearrowright X$ = set

$S \subseteq G$

$\forall s \in S \cup S^{-1}$: $X_s \subseteq X$

⓪ $p \in X \setminus \bigcup_s X_s$

and ① $s \cdot p \in X_s$ $\forall s \in S \cup S^{-1}$

② $s \cdot X_t \subseteq X_s$ $\forall t \neq s^{-1}$

↑ Meier says $\subsetneq$???

Then: $\langle S \rangle \cong F_S$

$\langle S \rangle$ means subgp gen by S.

F_S = free gp on S.

Distinctions from P.P.L. I :

① X_s's not disjoint

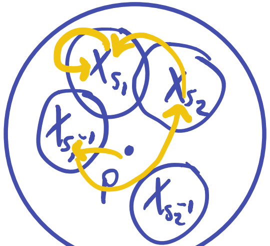
(replaced with existence of p)

② Only need $s^k \cdot X_t \subseteq X_s$ $k=1$.

(replaced with $s \cdot X_s \subseteq X_s$).

Pf. Look where p goes.

$s_1 \cdot s_1 \cdot s_2 \cdot s_1^{-1}$



3.4 Free gps & actions on trees

Thm. If a group acts freely on a tree, then it is free.

Pf #1 Say $G \curvearrowright T$ = tree

Let F = fund dom.

Call the $g \cdot F$'s tiles.

$S = \{g \in G : g \cdot F \cap F \neq \emptyset\}$

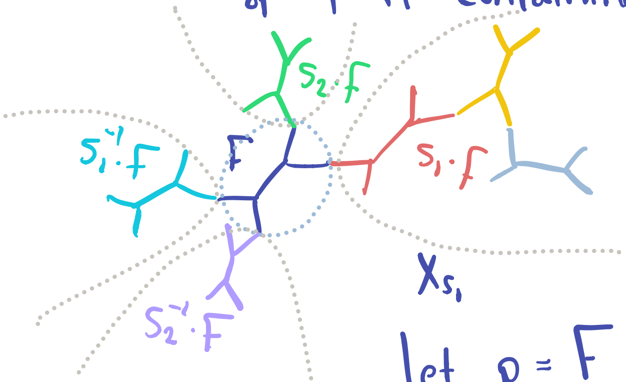
Previous thm $\Rightarrow$ S generates G.

To show: S generates a free gp.

Ping pong!

$X = \{\text{tiles}\}$

$X_s = \{\text{tiles that lie in component of } T \setminus F \text{ containing } s \cdot F\}$



Let $p = F$.

Freeness $\Rightarrow$ $s_i \cdot F \neq F \ \forall i$

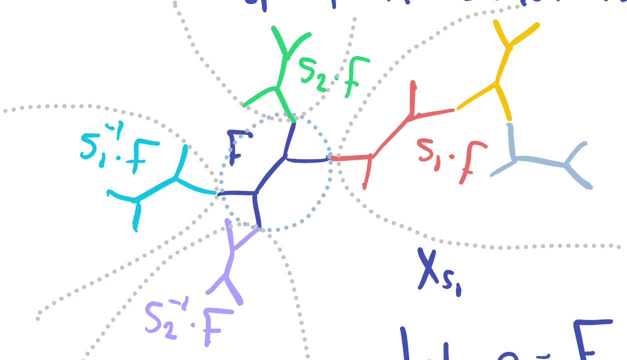
$\Rightarrow$ ⓪ in P.P.L. II.

① in PPL is by defn.

Remains to check ②.

$X = \{\text{tiles}\}$ ← tile in X_{s_2}

$X_s = \{$tiles that lie in component of $T \setminus F$ containing $s\cdot F\}$



Let $\rho = F$.

Freeness $\Rightarrow s_i\cdot F \neq F \quad \forall i$

$\Rightarrow$ (0) in P.P.L. II.

(1) in PPL is by defn.

Remains to check (2).

For (2) we'll do: $s_1\cdot X_{s_2} \subseteq X_{s_1}$

Consider the seq. of adjacent tiles:

$$s_1^{-1}\cdot F \longrightarrow F \longrightarrow s_2 F \longrightarrow \text{rest of } X_{s_2}$$

Apply s_1:

$$F \longrightarrow s_1\cdot F \longrightarrow \underbrace{s_1 s_2 F \longrightarrow \text{rest of } s_1\cdot X_{s_2}}_{s_1\cdot X_{s_2} \subseteq X_{s_1}}$$

since T is a tree! $\square$

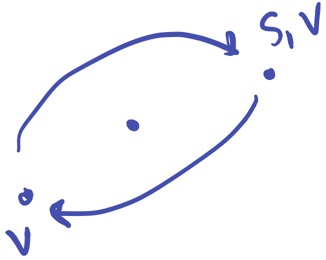
Tricky Special case: $s_1 \cdot X_{s_1} \subseteq X_{s_1}$

$$s_1^{-1} \cdot F \longrightarrow F \longrightarrow s_1 F \longrightarrow \text{rest of } X_{s_1}$$

Apply s_1:

$$F \to s_1 \cdot F \to \underbrace{s_1 s_1 F \to \text{rest of } s_1 \cdot X_{s_1}}_{s_1 \cdot X_{s_1} \subseteq X_{s_1}}$$

since T is a tree!



Fine if $s_1 s_1 F \neq F$

i.e. $s_1^2 = \mathrm{id}$.

Lemma from last time:

$\mathbb{Z}/2$ does not act freely on a tree.

$\Longrightarrow$ any gp with elt of order 2 does not act freely on a tree.

Is this page needed???

Cor. Subgroups of free gps are free.

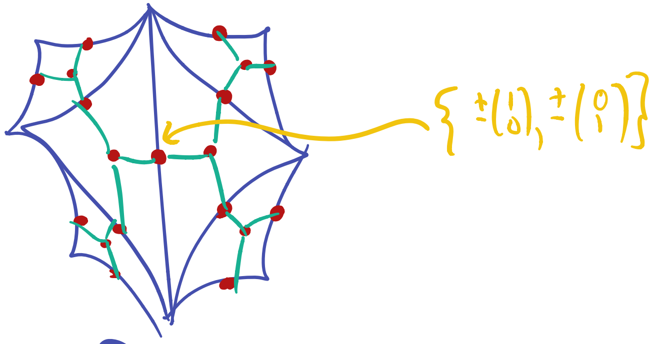
Pf. F = free gp

$F \circlearrowright T$ some tree (Cayley graph) freely.

Any subgp inherits a free action.

Apply the theorem. $\square$

Example $SL_2(\mathbb{Z})[m]$ is free $m \geq 3$.



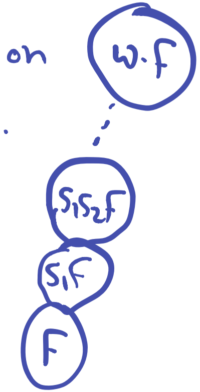
Check freeness.

Matrices fixing center vertex:

$$\pm I \qquad \pm\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

etc.

Thm. If a group acts freely on a tree, then it is free.



Pf #2 (OHGGT)

$G \curvearrowright T$ freely.

F = fund dom

$\rightsquigarrow S = \{s \in G : s\cdot F \cap F \neq \emptyset\}$.

Take a freely reduced word in S.

$w = s_1 \cdots s_k \qquad s_i \in S$.

Want: $w \neq \mathrm{id}$.

Will show: $w \cdot F \neq F$.

Will show

$$F,\ s_1\cdot F,\ s_1s_2\cdot F,\ \ldots,\ s_1\cdots s_k\cdot F$$

is a non-backtracking sequence of adjacent tiles.

Since T is a tree this implies

$$s_1\cdots s_k\cdot F \neq F.$$

Check $s_1\cdots s_i\cdot F$ adjacent, and not equal to $(s_1\cdots s_i)s_{i+1}\cdot F$

$s_{i+1}\cdot F$ adjacent to F (not equal F by freeness)

Apply $s_1\cdots s_i$ to both. $\square$

So: $\left\{\begin{matrix}\text{non backtracking}\\ \text{paths of tiles}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{freely red.}\\ \text{words in } S\end{matrix}\right\}$.

Ping Pong Lemma II

Lemma 3.10

Have $G \curvearrowright X$ = set

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$\forall s \in S \cup S^{-1} : X_s \subseteq X$

(0) $p \in X \setminus \bigcup_s X_s$

and (1) $s \cdot p \in X_s \quad \forall s \in S \cup S^{-1}$

(2) $s \cdot X_t \subseteq X_s \quad \forall t \neq s^{-1}$

Meier says $\subsetneq$???

Then: $\langle S \rangle \cong F_S$

After class, we decided that we need to assume S has no elts of order 2.

Example: $\{\pm 1\} = \mathbb{Z}/2 \curvearrowright \{\pm 1\}$

$S = \{-1\} \quad p = +1 \quad -1 \cdot p = -1$

$X_{-1} = \{-1\}$

(2) is vacuous here!

Pf of Thm is ok because we had a lemma about elts of order 2.

Noah suggested an alternate fix where we remove $t \neq s^{-1}$. Not sure if this version has any application