Announcements Feb 25

- Cameras on
- Abstracts Fri Feb 26 Gradescope (Team submissions)
- HW 6 due Thu
- Take home midterm March 4
- No office hour Fri this week.
- Regular office hours Tue 11, appt
- Ask for help on HW!

Today

- Ping pong lemma
- Free actions on trees $\leftrightarrow$ free groups
- Free actions on edges of trees $\leftrightarrow$ free products

Ping Pong Lemma II

Lemma 3.10

Have $G \curvearrowright X$ = set ✓

$S \subseteq G$ ✓

$\forall s \in S \cup S^{-1}$: $X_s \subseteq X$ ✓

⓪ $p \in X \setminus \bigcup_s X_s$ ✓

and ① $s \cdot p \in X_s \quad \forall s \in S \cup S^{-1}$ ✓

② $s \cdot X_t \subseteq X_s \quad \forall t \neq s^{-1}$

↑ Meier says $\subsetneq$???

Then: $\langle S \rangle \cong F_S$

After class, we decided that we need to assume S has no elts of order 2.

Example: $\{\pm 1\} = \mathbb{Z}/2 \curvearrowright \{\pm 1\}$

$S = \{-1\}$ $\quad p = +1 \quad -1 \cdot p = -1$

$X_{-1} = \{-1\}$

② is vacuous here!

Pf of Thm is ok because we had a lemma about elts of order 2.

Noah suggested an alternate fix where we remove $t \neq s^{-1}$. Not sure if this version has any application

3.4 Free gps acting on trees

Thm. If a group acts freely on a tree, it is free.

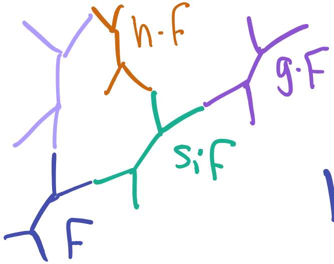
Pf #2 Let $G \curvearrowright T$ = tree freely.

F = fund. dom.

$\rightsquigarrow S = \{g : g \cdot F \cap F \neq \emptyset\}$ is a gen set $\Rightarrow \langle S \rangle = G$

Call the $g \cdot F$ <u>tiles</u>

The action preserves the tiling.



Want: no relations among S.

Let $w = s_1 \cdots s_k$ freely red. word in S / elt of G

Want $w \cdot F \neq F$ ($\Rightarrow w \neq id$)

treeness $\rightarrow$ True because the word w gives... a non-backtracking path of tiles from F to $w \cdot F$:

$F, s_1 F, s_1 s_2 \cdot F, \ldots, \underbrace{s_1 \cdots s_k \cdot F}_{w \cdot F}$

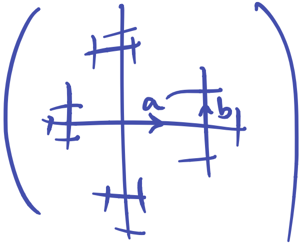
Indeed $s_1 \cdots s_{i+1} \cdot F$ adj to $s_1 \cdots s_i \cdot F$.

Freeness $\longrightarrow$ Apply $s_1 \cdots s_i$ to $F, s_{i+1} \cdot F$ $\square$

Sudipta Kolay.

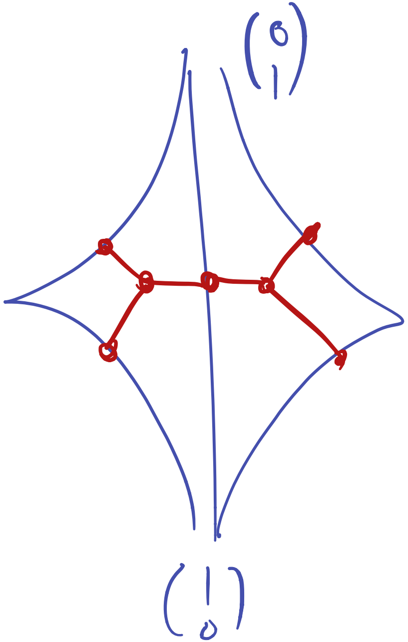
Examples

1. $\mathbb{Z} \curvearrowright$ 

2. Sym^+ 

3. $SL_2\mathbb{Z}[m] \curvearrowright$ Farey tree.

freely $m \geq 3$.



One partic. vertex of Farey tree:

$\left\{ \pm\binom{1}{0}, \pm\binom{0}{1} \right\}$ ← the elts of $SL_2\mathbb{Z}$ taking this vertex to itself: $\pm I, \pm\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

of these, only I is $\equiv I \bmod m$

$m \geq 3$.

3.6 Free products of groups

A, B groups

a word in $A \amalg B$ is freely reduced if alternates between nontrivial elts of A & B, eg:

$$a_1 b_1 a_2 b_2 a_3$$

$$A * B = \{\text{freely red. words in } A \amalg B\}$$

group op: concat & reduce.

$$(a_1 b_1)(b_2 a_2) = a_1 (b_1 b_2) a_2$$

$$a_1 b_3 a_2$$

$$(b_3 = b_1 b_2 \text{ in } B)$$

$$\boxed{\begin{array}{l} \mathbb{Z} \cup \mathbb{Z} = \mathbb{Z} \\ \mathbb{Z} \amalg \mathbb{Z} : \text{2 copies of } \mathbb{Z}. \end{array}}$$

Prop. $A * B$ is well-defined: any word can be reduced to a unique freely red. word. Just like for F_n.

Examples

① $\mathbb{Z} * \mathbb{Z} \cong F_2$

$$(5\ 7\ {-3}\ 10)(-1\ 1)$$

$$= 5\ 7\ {-3}\ 9\ 1$$

$$(x^5 y^7 x^{-3} y^{10})(y^{-1} x)$$

$$= x^5 y^7 x^{-3} y^9 x$$

② $\mathbb{Z}/2 * \mathbb{Z}/2 \cong D_\infty$

alternating words in $1, 1$

" " " a, b

$\cong \langle a, b \mid a^2 = b^2 = id \rangle$

③ $\mathbb{Z}/2 * \mathbb{Z}/3 \cong ?$

$\cong \langle a, b \mid a^2 = b^3 = id \rangle$

Some true things

① $A, B \leq A * B$

② $A * B \twoheadrightarrow A$ (or B)

kernel : B.

③ $A * B \longrightarrow A \times B$

kernel is free group (next time?)

e.g. $D_\infty \longrightarrow \mathbb{Z}/2$

word length mod 2

kernel : $\mathbb{Z}$

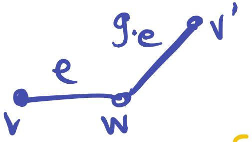
If $G \curvearrowright T$ denote stabilizer of v by G_v

free

Thm. If $G \curvearrowright T$ freely & transitively on edges and fund dom $F = v \overset{e}{\bullet\!\!-\!\!-\!\!\bullet} w$ and v, w in distinct G-orbits. Then $G \cong G_v * G_w$.

Pf. Step 1. G is gen by G_v & G_w.

$$S = \{ g : g \cdot F \cap F \neq \emptyset \}$$



$g \cdot e \in G_w$.

since v, w in distinct orbits.

Step 2. Take a freely red word in $G_v \amalg G_w$

$$w = a_1 b_1 a_2 b_2 \cdots a_k b_k$$

To show $w \neq id$ or $w \cdot F \neq F$.

Like last time:

$F, a_1 F, a_1 b_1 F, \ldots$

is a nonbacktracking path.

e.g. 

$\square$

tree

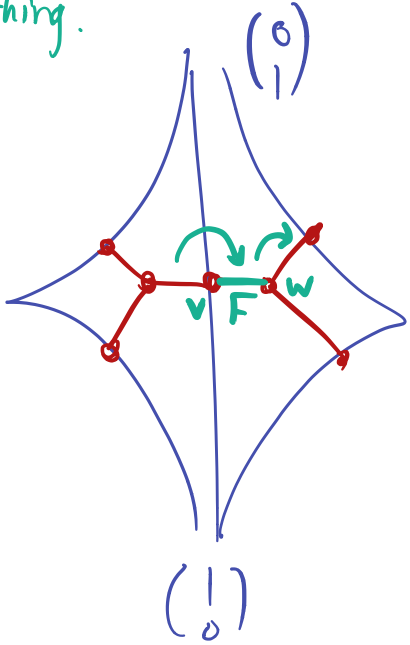
almost! $-I$ fixes everything.

Thm. If $G \circlearrowright T$ freely & transitively on edges

and fund dom $F = \overset{v}{\bullet}\!\xrightarrow{\;e\;}\!\overset{w}{\bullet}$

and v, w in distinct G-orbits.

Then $G \cong G_v * G_w$.



__Application__ $SL_2\mathbb{Z} \circlearrowright$ Farey tree

$$PSL_2\mathbb{Z} = SL_2\mathbb{Z}/\pm I$$
$$\cong \mathbb{Z}/2 * \mathbb{Z}/3.$$

$\square$