MATH 4803-MAR

Intro. Geometric Group Theory

Spring 2021 GATECH

Basic Info

Canvas → Piazza → 1st HW

Canvas → Grades

Canvas → Gradescope

Canvas → Web page

Web page → Calendar → HW

Calendar → Notes

Resources/Refs

Syllabus, Final Project

Assessments

10% Participation. (Piazza)

30% HW

30% Midterm

30% Final project

What is GGT?

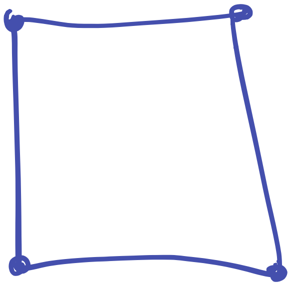
Groups are collections of symmetries of geometric objects

Use the geometry to learn about the algebraic properties of the group.

A group:

	e	r	r^2	r^3	f	rf	r^2f	r^3f
e	e	r	r^2	r^3	f	rf	r^2f	r^3f
r	r	r^2	r^3	e	rf	r^2f	r^3f	f
r^2	r^2	r^3	e	r	r^2f	r^3f	f	rf
r^3	r^3	e	r	r^2	r^3f	f	rf	r^2f
f	f	r^3f	r^2f	rf	e	r^3	r^2	r
rf	rf	f	r^3f	r^2f	r	e	r^3	r^2
r^2f	r^2f	rf	f	r^3f	r^2	r	e	r^3
r^3f	r^3f	r^2f	rf	f	r^3	r^2	r	e

It is D_4



A group:

$SL_2(\mathbb{Z}) = \left\{ \begin{array}{l} 2\times 2 \text{ integer} \\ \text{matrices of} \\ \det = 1 \end{array} \right\}$

examples: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Lin alg: eigenvals, eigenvectors, ...
lin transf. of $\mathbb{R}^2$

Group theory:

generators?

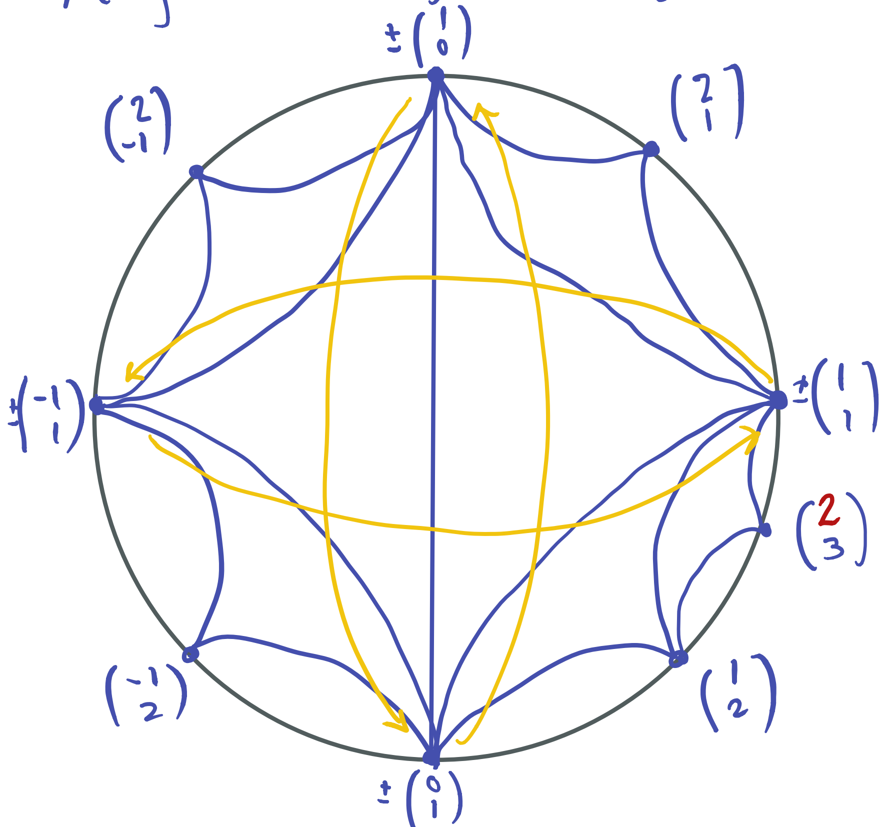
relations?

torsion?

subgroups?

quotients?

A geometric object: Farey graph



Not-obvious but true:

- all primitive integer vectors are vertices. not a multiple
- each $A \in SL_2(\mathbb{Z})$ gives a symmetry of the graph.

Check: if v, w connected by edge then Av, Aw connected by edge

$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ takes $\begin{pmatrix}1\\1\end{pmatrix}$

to $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

Rotation by π.

To do the check, prove:

$\begin{pmatrix} p \\ q \end{pmatrix}$ & $\begin{pmatrix} r \\ s \end{pmatrix}$ are connected by an edge

$$\Longleftrightarrow \det\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1.$$

Check: if v, w connected by edge then Av, Aw connected by edge

$$A\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \left(A\begin{pmatrix} p \\ q \end{pmatrix} \; A\begin{pmatrix} r \\ s \end{pmatrix} \right)$$

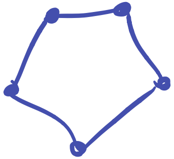
If $\begin{pmatrix} p \\ q \end{pmatrix} \bullet\!\!-\!\!\bullet \begin{pmatrix} r \\ s \end{pmatrix}$ then $\det\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1$

then $\det A\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \pm 1$ then $A\begin{pmatrix} p \\ q \end{pmatrix} \bullet\!\!-\!\!\bullet A\begin{pmatrix} r \\ s \end{pmatrix}$

Overview of Course

Chap 1. Cayley graph

$G \longleftrightarrow$ graph



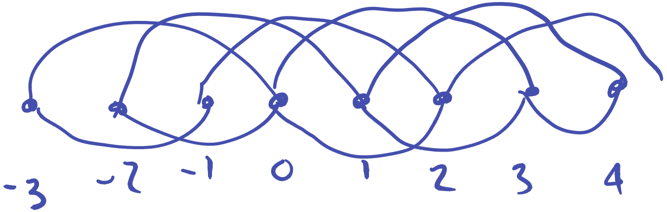
$\mathbb{Z}/5\mathbb{Z}$

0 ends

2 ends

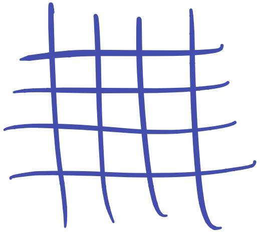
$\mathbb{Z}$

2 ends.

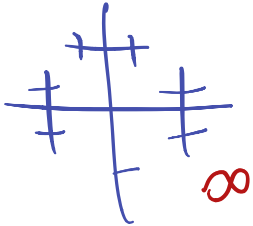


$\mathbb{Z}$ with gen set $\{3,2\}$

1 end



$\mathbb{Z}^2$

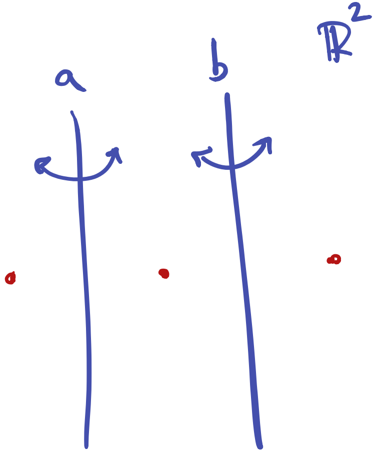


∞ ends

Chap 2 Coxeter gps

= groups gen by reflections

example



ab

Chap 3 Groups acting on trees

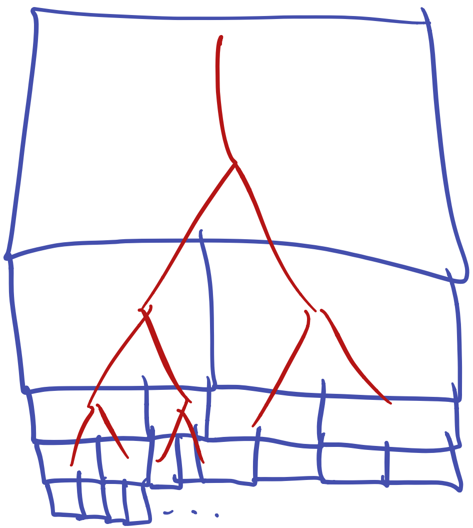
Free gps

F_n = gp with n gens & no relations

- Groups acting (freely) on trees $\longleftrightarrow$ free gps
- $F_3 \leq F_2$
- $F_\infty \leq F_2$

Chap 4 Baumslag-Solitar gps

see pic on web site.



hyperbolic plane. "treelike"

Chap 5 Word problem

Given a product of gens, is it id in gp.

Chap 8 Lamplighter gp

Chap 10 Thompson's gp.

Chap 11 Large scale properties

Thm. Every gp has 0, 1, 2, or ∞ many ends.