# Announcements Jan 19

- Cameras on in class
- 1st HW assigned Thu, due Tue 3:30. Gradescope
- Lecture notes/HW posted on web site.
- Groups / topics due Feb 5
- Office hours Tue 11-12, Fri 2-3, appt

Q. How many symmetries does a cube have?
tetrahedron, icosahedron, ... ?

# Groups

$G$ set

$G \times G \longrightarrow G$ mult.

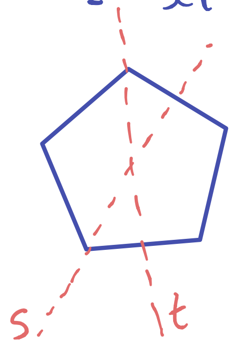
w/ id
inv.
assoc.

example: symmetries of... anything.

# Examples of finite groups

① Dihedral group $D_n$

= set of symmetries of $n$-gon




s, t are generators.

Since st is a rotation

relations:

$$s^2 = t^2 = id.$$

$$(st)^n = id.$$

This is a presentation for $D_n$:

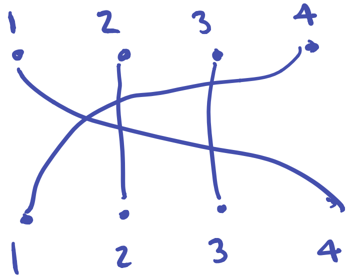
generators → $\langle s, t \mid s^2 = t^2 = (st)^n = id \rangle$ ← relators

$s = s~~(st)^n~~$

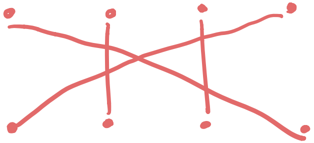
## ② Symmetric group.

$S_n$ = set of permutations of $\{1, \dots, n\}$.

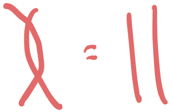
generators: $(i \;\; i+1) = \tau_i$




$(3\;4)(2\;3)(1\;2)(2\,3)(3\;4)$



Generators: $\tau_1, \dots, \tau_{n-1}$



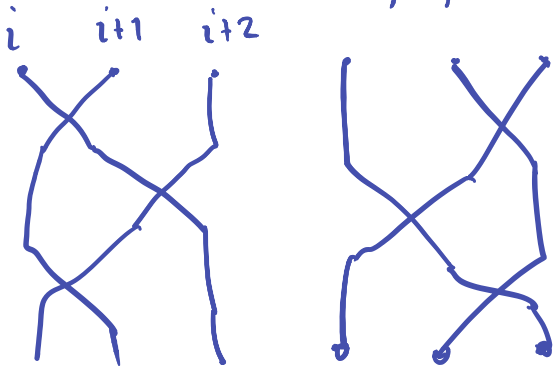

Relations: $\tau_i^2 = \mathrm{id}$.

$\tau_i \tau_j = \tau_j \tau_i \quad |i-j| > 1$

$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$

$i = 1, \dots, n-2$

These give a presentation for $S_n$!

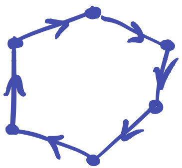



$\tau_i \tau_{i+1} \tau_i$ $\qquad$ $\tau_{i+1} \tau_i \tau_{i+1}$

## ③ Finite cyclic gps

$\mathbb{Z}/n\mathbb{Z}$

What is it the symmetries of?



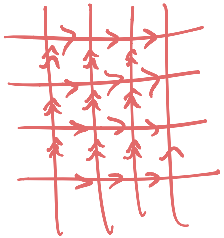
n-gon.

Presentation?

$$\langle a \mid a^n = id \rangle$$

## ④ Trivial group

$\langle 1 \rangle$ or $\langle a \mid a \rangle$



## Examples of Infinite groups

① $\mathbb{Z}$

$\langle a \rangle$

$a$, $a^2$, $a^3$

$a^{-1}$, $a^{-2}$, $a^{-3}$

② $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(a,b) : a,b \in \mathbb{Z}\}$

$\langle a, b \mid ab = ba \rangle$

$a = (1,0)$

$b = (0,1)$

$aba = a^2 b = (2,1)$

③ $SL_n\mathbb{Z}$ = { n×n integer matrices with det = 1 }

What is this the symmetries of?

Presentation?

Harder!

# ④ Free groups

no relations.

$F_2 = \{$ freely reduced finite words in $a, b\}$

word: finite sequence of $a, b, a^{-1}, b^{-1}$

freely reduced: no $aa^{-1}$, $a^{-1}a$, $bb^{-1}$, $b^{-1}b$

Multiplication: concatenate, then freely reduce

e.g. $aba^{-1} \cdot ab = abb$

Check this is a group.

id = empty word.

inverse = reverse & invert letters

e.g. $(abab)^{-1} = b^{-1}a^{-1}b^{-1}a^{-1}$

assoc. ✓

An issue: different reductions lead to same reduced word.

e.g. $aa^{-1}bb^{-1}$

or $b^{-1}aa^{-1}bb^{-1}$

Presentation: $\langle a, b \mid \rangle$

So...

$\mathbb{Z} \cong F_1$

& $F_0$ = trivial group.

Later in the class:

In $SL_2(\mathbb{Z})$:

$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

generate a free group.

so: $a^5 b^7 a^{-1} b^{-4} a \neq id$.

Next time: Free gps are important because every countable group is a quotient of a free group.