Announcements Jan 21

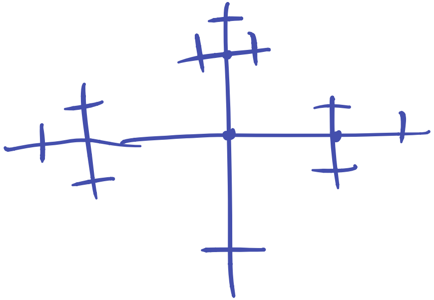
- Please turn cameras on
- HW1 due Tue 3:30 (I need to set up Gradescope)
- HW1 Lecture notes posted on web site. Need to add a reading prompt
- Groups/topics due Feb 5
- Office hours Fri 2-3, Tue 11-12, by appt.

Examples of groups

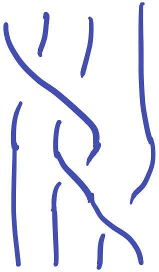
D_n, S_n, $\mathbb{Z}/n$

$\mathbb{Z}^n$, $SL_n\mathbb{Z}$, F_n

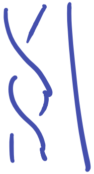
What is F_n the symmetries of F_2?



Braid groups B_n ← # of strands

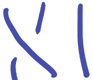
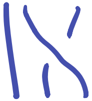
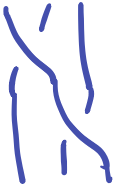
elements:  up to isotopy: 

in B_4

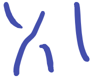


a^2

multipl: concatenation

B_3  $\cdot$  $=$ 

a $\quad$ b $\quad$ ab



a^{-1}

Internal presentations

$\langle S \mid R \rangle$ is an internal presentation of G

if ① S is a generating set for G

② If two words in $S \cup S^{-1}$ are equal in G, they differ by a finite seq. of elements of $R \cup \{ss^{-1} : s \in S \cup S^{-1}\}$

(replacing one side of an equality with another)

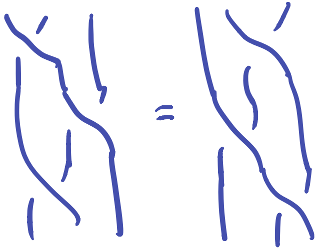
Fact. Every group has one: $S = G$

R = every possible equality.

<u>Example</u> $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1$

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$

σ_i



Homomorphisms

$f: G \to H$

$f(ab) = f(a)f(b)$

Injective homomorphisms

"putting one group into another as a subgp"

- $\mathbb{Z}/n \to D_n$ rotations.
- $\mathbb{Z}/2 \to D_n$ reflection.
- $\mathbb{Z} \to F_2$

  $1 \mapsto a$

Which groups have inj homoms to B_n?

$\mathbb{Z}$ $\quad 1 \mapsto \sigma_1$ ✓

$\mathbb{Z}^2$ $\quad (1,0) \mapsto \sigma_1$, $(0,1) \mapsto \sigma_3$ $\quad$ assuming $n \geq 4$

$\mathbb{Z}/2$ $\quad$ Does B_n have an elt of order 2?

No.

F_2 $\quad a \mapsto \sigma_1^2$, $b \mapsto \sigma_2^2$

Non-injective homoms

"forgetting (wisely)"

$\mathbb{Z} \to \mathbb{Z}/2$ even/odd

$\mathbb{Z} \to \mathbb{Z}/10$ 1's digit

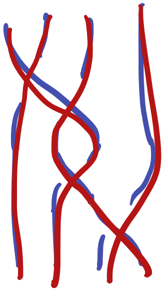
$D_n \to \mathbb{Z}/2$ flip?

$F_2 \to \mathbb{Z}^2$ exponent sum on a & b

$a \mapsto (1,0)$

$b \mapsto (0,1)$

$B_n \to S_n$



Normal subgps $N \leq G$

$$gNg^{-1} = N \quad \forall\, g \in G$$

$$\left\{ \begin{array}{c} \text{kernels} \\ \text{of } G \to \square \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{normal subgps} \\ \text{of } G \end{array} \right\}$$

First Isomorphism Thm

If $f: G \to H$ surj homom. with kernel K

Then $H \cong G/K$.

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Surj. homoms

$\mathbb{Z} \to \mathbb{Z}/2$

$\mathbb{Z} \to \mathbb{Z}/10$

$D_n \to \mathbb{Z}/2$

$F_2 \to \mathbb{Z}^2$

$a \mapsto (1,0)$

$b \mapsto (0,1)$

$B_n \to S_n$

$\xRightarrow{\text{1st Isom. Thm}}$

$\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2$

$D_n/\text{rotations} \cong \mathbb{Z}/2$

$D_n/\mathbb{Z}/n \cong \mathbb{Z}/2$

$F_2/F_2' = \mathbb{Z}^2$

commutator subgp.

$B_n/PB_n \cong S_n$

External presentation

$\langle S \mid R \rangle$

S = set

elts of R: equalities between words in $S \cup S^{-1}$ and id.

Not $ab = ba$, but $aba^{-1}b^{-1} = \mathrm{id}$

free gp on S

We obtain a group: $F(S) / \langle\langle R \rangle\rangle$

normal closure of R
= smallest normal subgp of $F(S)$ containing R.
= subgp of $F(S)$ gen. by elts of R & their conjugates.

<u>HW</u>. Internal & External presentations are equivalent.

<u>Consequence</u>. Every gp is a quotient of a free group.

$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} = \mathrm{id} \rangle = F^2 / \langle\langle \; \rangle\rangle$