Announcements Jan 26

- Cameras on for class
- HW1 due Thu 3:30
- Groups/topics due Feb 5
- Office hours Fri 2-3, Tue 11-12, appt

How many symmetries does a cube have?

\langle a, b, c, d \mid ab^{-1}c^{-1}a^{-1}d = 1 \rangle
Symmetries

\( X = \text{math object} \)

\( \text{Sym}(X) = \{\text{symmetries of } X\} \)

\( \text{group under composition.} \)

examples

\[
\begin{array}{c|c}
X & \text{Sym}(X) \\
\hline
\text{regular n-gon} & D_n \\
\{1, \ldots, n\} & S_n \\
n\text{-dim vector space} & \text{GL}_n \mathbb{F} \\
\text{over } \mathbb{F} & \\
\end{array}
\]

\( \mathbb{R}^2 \text{ as a vector space} \)

\( \text{Euclidean 2-space} \)

\( \text{topological space} \)

\( \text{group } G \)

\( \text{Aut}(G) \)

\( \text{ Aff}(\mathbb{R}^2) \text{ line-preserving} \)
**Actions**

An action of a group $G$ on a math object $X$ is a homomorphism

$$G \to \text{Sym}(X)$$

or a map

$$G \times X \to X$$

with

$$e \cdot x = x \quad \forall x \in X$$

$$g \cdot (h \cdot x) = (gh) \cdot x \quad \forall g, h \in G, x \in X$$

and the restriction

$$g \times X \to X$$

is in $\text{Sym}(X)$ $\forall g \in G$

**Examples**

- $D_n \subseteq \text{C}$ $n$-gon (filled in or not)
- $\mathbb{R}^2$
- $\{\text{vertices of } n\text{-gon}\}$
- $\{\text{diagonals of } n\text{-gon}\}$
- $\text{SL}_2 \mathbb{Z} \subseteq \text{C}$ $\mathbb{R}^2$ as a vector space
- $\{\text{vectors in } \mathbb{R}^2\}$
- $\{\text{primitive vectors} \ldots \}$
- $\text{C}$ $\text{Farey graph}$
Two vocab words:
1. If have $G \curvearrowright X$, say $G$ is represented by symmetries of $X$.
2. An action is faithful if $G \rightarrow \text{Sym}(X)$ is injective.

Cayley's Thm
Every group can be represented as a group of permutations

Rephrase: there is $G \hookrightarrow \text{Sym}(X)$, $X$ a set.

If $X = G$ as a set. $\rightarrow \text{Sym}(X)$ is a permutation group.
Given $g \in G$ need a permutation of $X = G$.

or $G \times G \rightarrow G$

$(g, h) \mapsto gh$

Need to check: defn of action. faithful
Examples

\[ G = \mathbb{Z}^2 \quad \text{and} \quad X = \mathbb{Z}^2 \]

The action of \((2,1)\) on \(\mathbb{Z}^2\)
**Graphs**

A graph $\Gamma$ is a set $V(\Gamma)$, a set $E(\Gamma)$, and a function $\text{Ends} : E(\Gamma) \rightarrow \{\{u,v\} : u,v \in V(\Gamma)\}$.

**Examples**

- $K_n = \text{complete graph on } n \text{ vertices}$
- $K_{m,n} = \text{complete bipartite graph}$
- Tree = connected graph with no cycles.
Symmetries of graphs

A symmetry of a graph $\Gamma$ is a pair of bijections

$\alpha : V(\Gamma) \rightarrow V(\Gamma)$

$\beta : E(\Gamma) \rightarrow E(\Gamma)$

preserving Ends function:

$\text{Ends}(\beta(e)) = \alpha \text{Ends}(e)$

Examples

$\text{Sym}(K_n) \cong \text{Sym}_n$

$\cong \text{Sym}(V(K_n))$

$\text{Sym}(K_{m,n}) \cong \text{Sym}_m \times \text{Sym}_n$

unless $m = n$.

How many symmetries? 12

$\text{deg} = 3 \rightarrow \text{Sym}_3 \times \mathbb{Z}/2$

$\text{deg} = 2$

permute
Graphs can have "decorations":

- directed edges
- labeled edges

"directed graph"

\[ \text{Sym}^+(\Gamma) = \{ \text{symmetries of } \Gamma \text{ preserving decorations} \} \]

Cayley's better theorem: Every group is faithfully represented as symmetries of a graph. (next time)
Cayley graphs

$G = \text{group}$

$S = \text{gen set}$.

Cayley graph for $G$ with respect to $S$ has vertices: $G$

edges: $g \rightarrow gs \quad g \in G, s \in S$

Examples:

1. $G = \mathbb{Z}/n, \quad S = \{1\}$
2. $G = \mathbb{Z}^2, \quad S = \{(1,0), (0,1)\}$