ANNOUNCEMENTS JAN 26

$$\langle a, b, c, d \mid a b^2 c^3 a^{-1} d = 1 \rangle$$

- Cameras on for class
- HW1 due Thu 3:30
- Groups/topics due Feb 5
- Office hours Fri 2-3, Tue 11-12, appt

How many symmetries does a cube have?

Symmetries

X = math object

$\mathrm{Sym}(X) = \{\text{symmetries of } X\}$

group under composition.

examples

X	$\mathrm{Sym}(X)$
regular n-gon	D_n
$\{1, \dots, n\}$	S_n
n-dim vector space over F	$GL_n F$

$\mathbb{R}^2$ as a vect sp	$GL_2 \mathbb{R}$
Euclidean 2-space	$\mathrm{Aff}(\mathbb{R}^2)$ $=$ line-preserving
topological space	$\mathrm{Homeo}(X)$
group G	$\mathrm{Aut}(G)$

Actions

An action of a group G on a math object X is a homomorphism

$$G \to \mathrm{Sym}(X)$$

G does something to X

Write: $G \circlearrowright X$

acts on

<u>or</u> a map

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g \cdot x$$

with

$$e \cdot x = x \quad \forall\, x \in X$$
$$g \cdot (h \cdot x) = (gh) \cdot x \quad \forall\, g, h \in G,\ x \in X$$

and the restriction

$$g \times X \longrightarrow X$$

is in $\mathrm{Sym}(X)$ $\forall\, g \in G$

Examples

$D_n \circlearrowright$ n-gon (filled in or not)

$\mathbb{R}^2$

{vertices of n-gon}

{diagonals of n-gon}

$SL_2\mathbb{Z} \circlearrowright \mathbb{R}^2$ as a vector space

$\circlearrowright$ {vectors in $\mathbb{R}^2$}

$\circlearrowright$ {primitive vectors...}

($\circlearrowright$ Farey graph.)

Two vocab words:

① If have $G \curvearrowright X$ say G is <u>represented</u> by symmetries of X.

② An action is <u>faithful</u> if $G \to \mathrm{Sym}(X)$ is injective.

<u>Cayley's Thm</u>

Every group can be represented faithfully as a group of permutations

Rephrase: there is $G \hookrightarrow \mathrm{Sym}(X)$ X: a set.

$g \neq id \longmapsto$ not id

<u>Pf.</u> Take $X = G$ as a set. $\rightsquigarrow$ $\mathrm{Sym}(X)$ is a permutation group.

Given $g \in G$ need a permutation of $X = G$.

or $G \times G \to G$

$(g, h) \longmapsto gh$

Need to check: defn of action. ✓

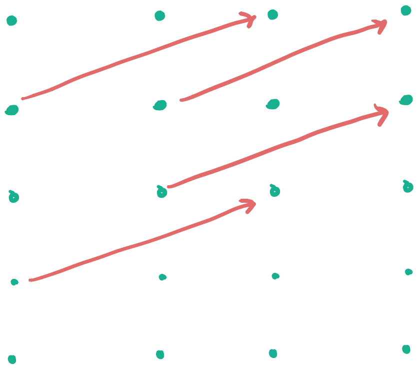
faithful

$\square$

Examples

$G = \mathbb{Z}^2$

$X = \mathbb{Z}^2$



The action of $(2,1)$ on $\mathbb{Z}^2$

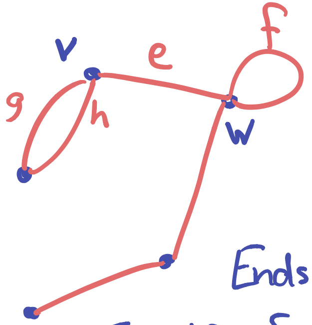
Graphs

A graph Γ is a set $V(\Gamma)$, a set $E(\Gamma)$,

vertices   edges

and a function

$$\mathrm{Ends} : E(\Gamma) \longrightarrow \{\{u,v\} : u,v \in V(\Gamma)\}$$

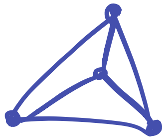


$\mathrm{Ends}(f) = \{w\}$

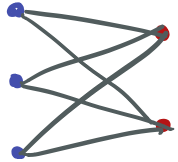
$\mathrm{Ends}(e) = \{v,w\}$

Examples

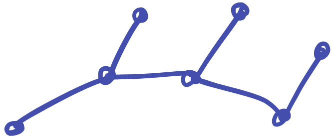
K_n = complete graph on n vertices



$K_{m,n}$ = complete bipartite graph ...



Tree = connected graph with no cycles.



Symmetries of graphs

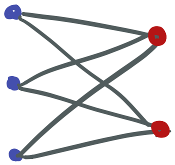
A **symmetry** of a graph Γ is a pair of bijections

$$\alpha : V(\Gamma) \longrightarrow V(\Gamma)$$

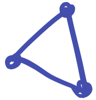
$$\beta : E(\Gamma) \longrightarrow E(\Gamma)$$

preserving Ends function:

$$\mathrm{Ends}(\beta(e)) = \alpha\, \mathrm{Ends}(e)$$



Examples



$$\mathrm{Sym}(K_n) \cong \mathrm{Sym}_n$$

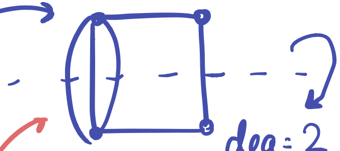
$$\cong \mathrm{Sym}(V(K_n))$$

$$\mathrm{Sym}(K_{m,n}) \cong \mathrm{Sym}_m \times \mathrm{Sym}_n$$

unless $m=n$.

How many symmetries? 12

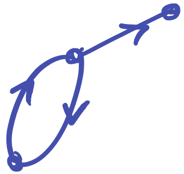
$\mathrm{Sym}_3 \times \mathbb{Z}/2$



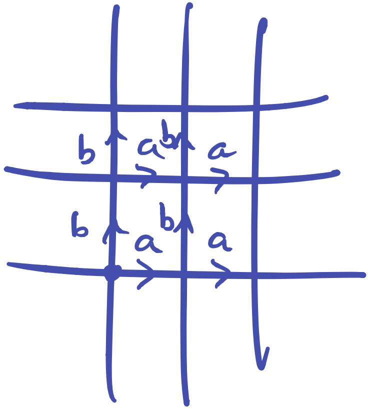
Graphs can have "decorations":

- directed edges

  "directed graph"



- labeled edges



$\mathrm{Sym}^+(\Gamma) = \{$ symmetries of Γ preserving decorations $\}$

Cayley's better theorm: Every group is faithfully rep. as symmetries of a graph. (next time)

Cayley graphs

G = group

S = gen set.

$\rightsquigarrow$ Cayley graph for G with respect to S

has vertices: G

edges: $g \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\, gs$ $\quad g \in G$, $s \in S$

Examples ① $G = \mathbb{Z}/n$ $\quad S = \{1\}$ ② $G = \mathbb{Z}^2$ $\quad S = \{(1,0), (0,1)\}$