Announcements Jan 28

RECORD

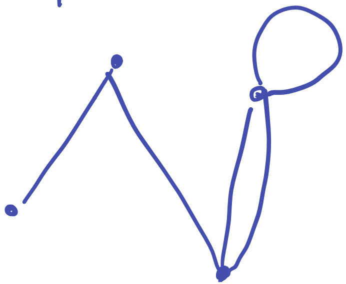
- Cameras on
- HW 2 due next week Thu 3:30
- Groups/topics due Feb 5
- Office hours Fri 2-3, Tue 11-12, appt
- Way-too-early course evals

Goals today:

- Given a graph, what are its symmetries?
- Given a group, what graph(s) does it act on?

Why? e.g. <u>Thm.</u> If $G \circlearrowright$ tree freely then G is free.

Last time: ① Graphs



exactly 2 symmetries.

Symmetries: permutation of $V(\Gamma)$, $E(\Gamma)$ respecting Ends.

② Actions

$G \circlearrowright X$ means: $G \longrightarrow \mathrm{Sym}(X)$

or

$G \times X \longrightarrow X$

$(g, x) \longmapsto g \cdot x$

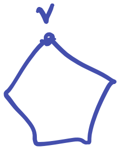
1.4 Orbits & stabilizers

Say $G \circlearrowright X$

$\mathrm{Stab}(x) = \{ g \in G : g \cdot x = x \}$

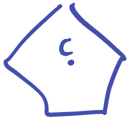
this is a subgroup

e.g. • $D_n \circlearrowright$ n-gon.



$\mathrm{Stab}(v) \cong \mathbb{Z}/2$

• $D_n \circlearrowright$ n-gon

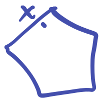


$\mathrm{Stab}(c) = D_n$

The action of G is <u>free</u> if $\mathrm{Stab}(x) = \{e\}$ $\forall x$

$\mathrm{Orb}(x) = \{ g \cdot x : g \in G \}$

eg. $D_n \circlearrowright$ n-gon.



$|\mathrm{Orb}(x)| = 2n$

$|\mathrm{Orb}(v)| = n$

$|\mathrm{Orb}(c)| = 1$

$\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}$

$\mathrm{Orb}(x) = \{g \cdot x : g \in G\}$

<u>Thm</u>. There is a bijection:

$$\mathrm{Orb}(x) \longleftrightarrow \text{left cosets of } \mathrm{Stab}(x)$$

given by

$$g \cdot x \longleftrightarrow g\,\mathrm{Stab}(x).$$

<u>Pf</u>. Subtlety: well-definedness.

But $g \cdot x = h \cdot x \iff h^{-1} g \cdot x = x$

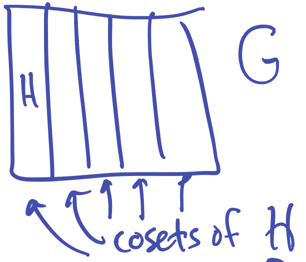
$\iff h^{-1} g \in \mathrm{Stab}(x) \iff g\,\mathrm{Stab}(x) = h\,\mathrm{Stab}(x)$ $\square$

<u>Cor</u> (Orbit-Stab Thm)

If $|G| < \infty$ & $G \curvearrowright X$ then

$$|G| = |\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)|$$

<u>Pf</u>. Lagrange's thm



<u>Example</u>. How many symmetries does a cube have?

$|G| = 3 \cdot 8 = 24$ rotations

or $6 \cdot 8 = 48$ all symmetries

<u>Cor</u>^{2} If $\mathrm{Stab}(x) = \{e\}$ then: $G \longleftrightarrow \mathrm{Orb}(x)$

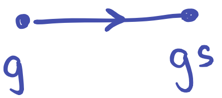
1.5 Cayley graphs

G $\quad S$ = gen set.

$\rightsquigarrow \Gamma_{G,S}$

$V(\Gamma_{G,S}) = G$

$E(\Gamma_{G,S})$:



$\forall\, g \in G$
$\quad s \in S$

Fact. $\Gamma_{G,S}$ is connected. (as undirected graph)

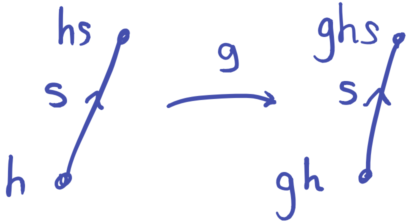
Why? Enough to show all vertices are connected by a path to e.
Given a vertex g, write as a product $g = s_1 \cdots s_n$ $\; s_i \in S$.

Fact. $G \circlearrowright \Gamma_{G,S}$ as a labeled, directed graph

On vertices: $g \cdot h = gh$

gp elt. vertex vertex

This extends to edges:

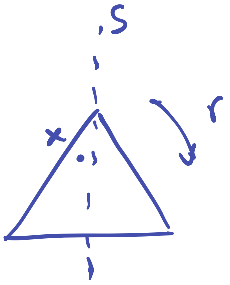


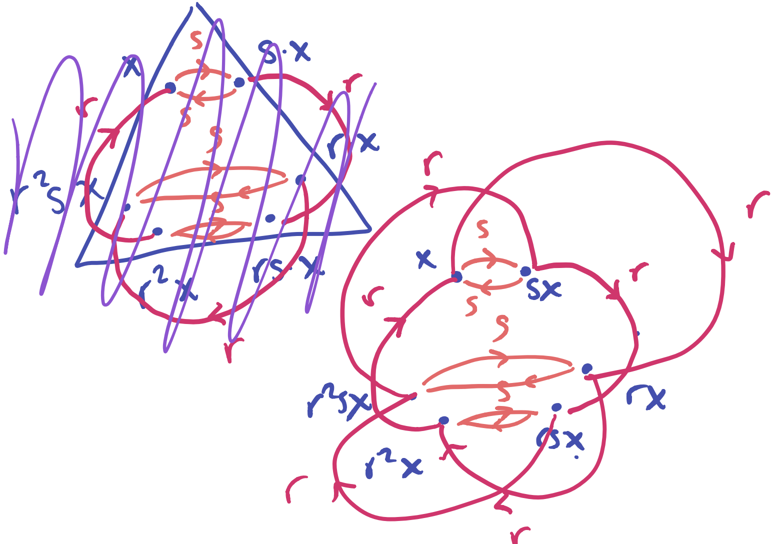
Fact. The action is faithful.

Examples

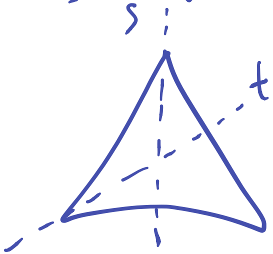
① $D_3 = Sym_3$

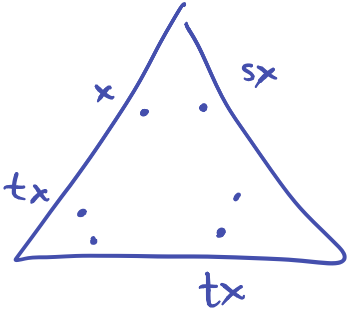
$S = \{r, s\}$





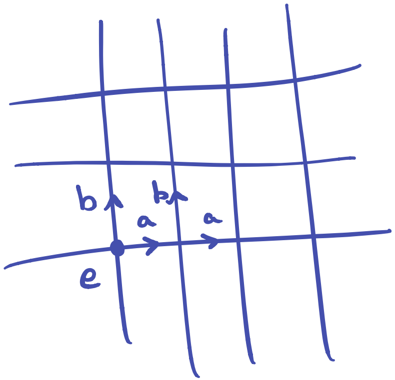
$D_3 = Sym_3$ $\quad S = \{s, t\}$





exercise...

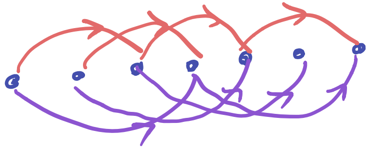
$\mathbb{Z}^2$



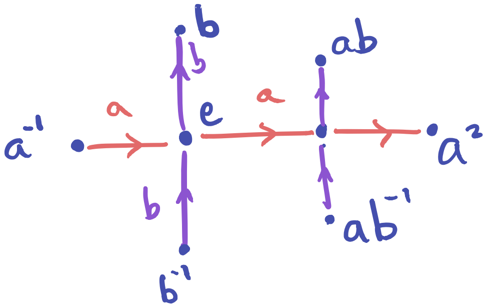
$S = \{a, b\}$, $a = (1,0)$, $b = (0,1)$

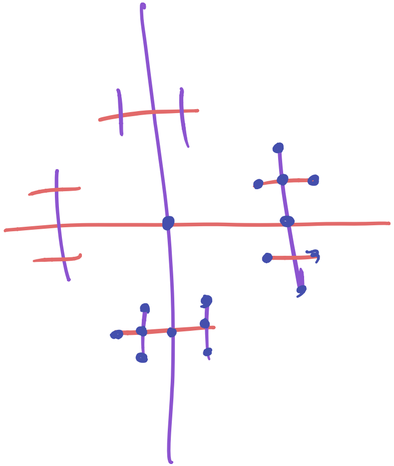
$\mathbb{Z}$ $\quad S = \{1\}$

or $\mathbb{Z}$ $\quad S = \{2, 3\}$



F_2 $\quad S = \{a, b\}$





Cycles in $\Gamma_{G,S}$

$\longleftrightarrow$ relations.