ANNOUNCEMENTS MAR 2

- Cameras on
- HW 6 due Thu
- Midterm Mar 4-11
- Office hours Fri 2-3, Tue 11-12, appt.

Today: Free products & trees
Free products are virtually free.

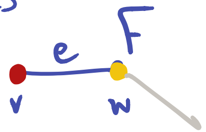
3.6 Free products

$A * B$

Thm. $G \curvearrowright T$ = tree.

freely, transitive on edges. redundant

2 orbits of vertices

fundamental domain 

Then $G \cong G_v * G_w$.

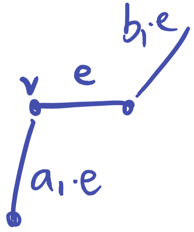
Pf. Step 1. $S = \{g \in G : g \cdot F \cap F \neq \emptyset\}$

$= G_v \cup G_w$

generates G

Step 2. Any freely red word $w = a_1 b_1 \cdots$

$a_i \in G_v \quad b_i \in G_w$



gives a nonbacktracking path from e and $w \cdot e$

the path is:

$e,\ a_1 \cdot e,\ a_1 b_1 \cdot e,\ \cdots$

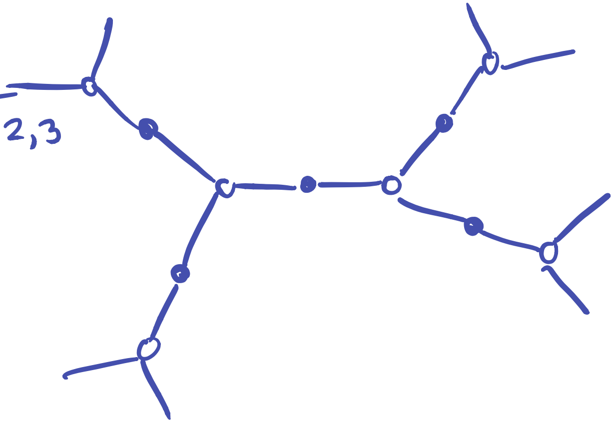
non back track $\Longrightarrow w \cdot e \neq e$

$\Longrightarrow w \neq \mathrm{id}.$ $\square$

Application: $PSL_2\mathbb{Z} \cong \mathbb{Z}/2 * \mathbb{Z}/3$

3.8 A converse

Thm 3.28 Say $A * B$ is a free prod. Then $\exists$ bipartite tree T and an action of $A*B$ satisfying the last theorem. If $|A|,|B| < \infty$ then $T = T_{|A|,|B|}$

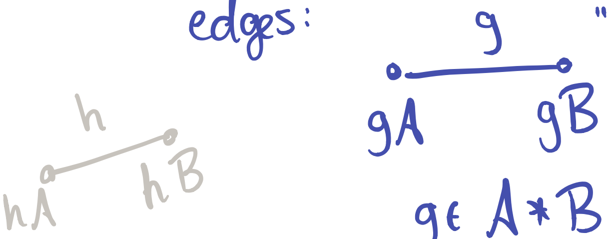


$T_{2,3}$

Pf. blue vertices : cosets of A

white vertices : cosets of B in $A*B$.

edges:



"g edge"

$g \in A*B$

Action: left mult

Q. When do g- & h-edges intersect?

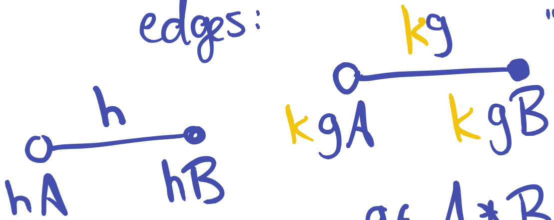
A. $g^{-1}h \in A$ or B.

Pf. blue vertices : cosets of A

white vertices : cosets of B

in $A*B$.

edges: "g edge"



$g \in A*B$

Check things :

① T is bipartite.

have A vertices
B vertices

because can't have $gA = hB$.

If $gA = hB$ then $h^{-1}gA = B$

but $id \in B \Rightarrow id \in h^{-1}gA$

$\Rightarrow h^{-1}gA = A$. But $A \neq B$.

② Action is free on edges. ✓

③ Two orbits of vertices ✓
(same as bipartiteness)

④ Transitively on edges ✓

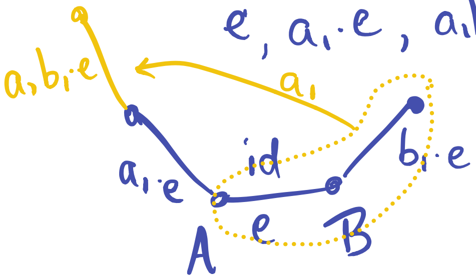
⑤ T is connected $\quad g \in A * B$

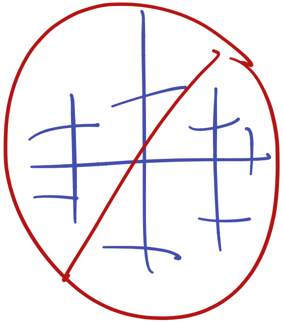
To connect id-edge e to g-edge:

write $g = a_1 b_1 \cdots$

the path of edges is

$e,\ a_1 \cdot e,\ a_1 b_1 \cdot e, \ldots$





⑥ T is acyclic.

Nonbacktracking paths

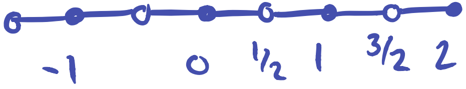
$\longleftrightarrow$ freely red. words

$\square$

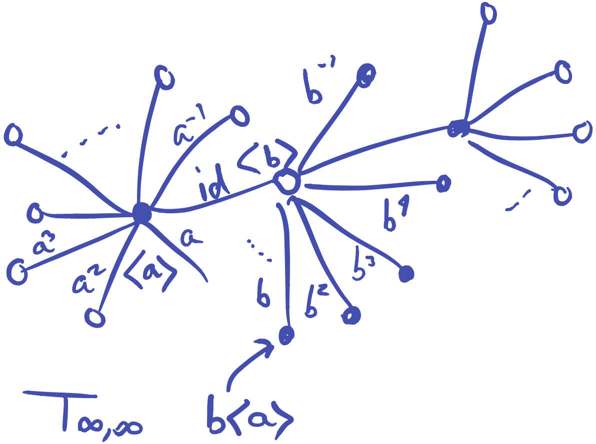
Examples

① $\mathbb{Z}/2 * \mathbb{Z}/2$

$\rightsquigarrow T_{2,2} =$



② $\mathbb{Z} * \mathbb{Z} \cong F_2 = \langle a, b \rangle$

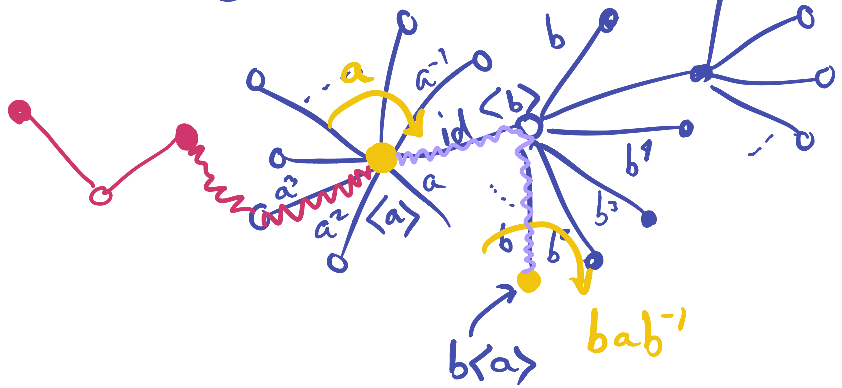


$T_{\infty,\infty}$

The stabilizer of the vertex A is A

<u>Prop</u>. The stabilizer of gA is gAg^{-1}

$$\underline{gAg^{-1}}\, gA = gA$$



<u>Poll</u>. Consider $\overset{\langle a, c\rangle}{\langle a, bab^{-1}\rangle} \leq F_2 \ldots$ is it free?

Yes. Same as proof at start of class...
A reduced word in a, c gives a non-back path (2 edges for each "syllable").

3.8 Thm
Let A, B be finite groups then $A * B$ is virtually free (it has a free subgp of finite index).

Pf. We'll prove more: kernel K of
$$A * B \to A \times B$$
is free. Kernel has index $|A \times B| < \infty$.

Make the tree T for $A * B$ as above.

Check K acts freely.

Stabilizers of edges in $A * B$, hence K, are trivial.

Nontrivial Stabilizers of vertices in $A*B$ are of form gag^{-1} $a \in A$, $a \neq id$.
which maps to $a \times id$ in $A \times B$
$\Rightarrow gag^{-1}$ not in K. $\square$

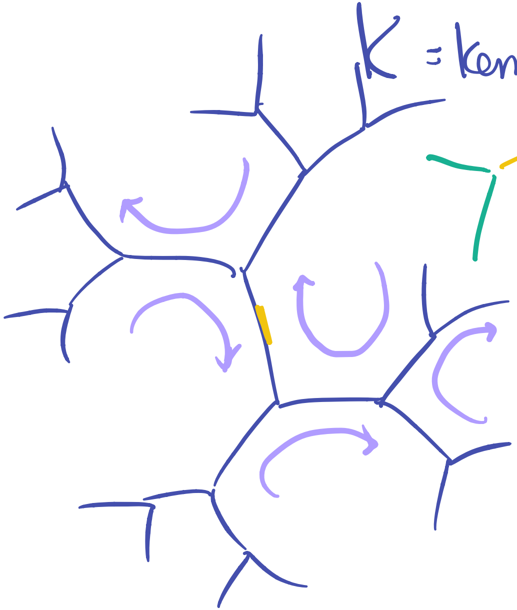
HW #20

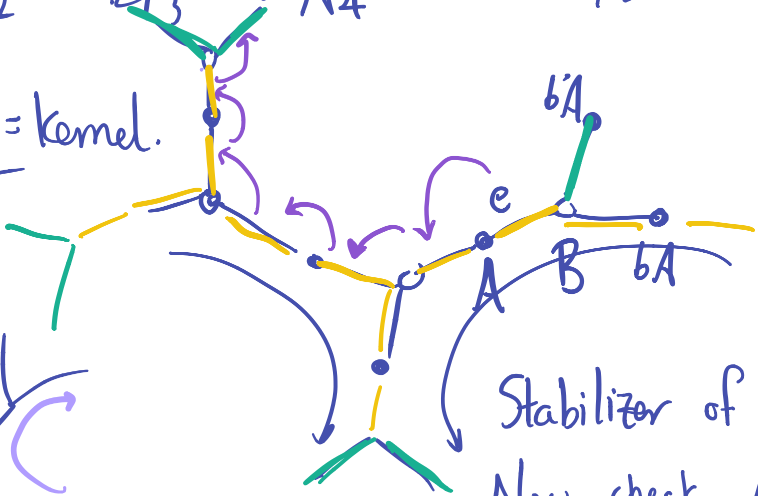
$\mathbb{Z}/2 * \mathbb{Z}/3 \to A_4$

A B

$\mathbb{Z}/2 * \mathbb{Z}/3 \circlearrowright T_{2,3}$

K = kernel.





Stabilizer of A is A.

Now check $A \cap K = \{id\}$.

For each vertex/~~edge~~, find stabilizer (by Prop), show intersection with K is $\{1\}$.

Generalizing to $A * B * C$.