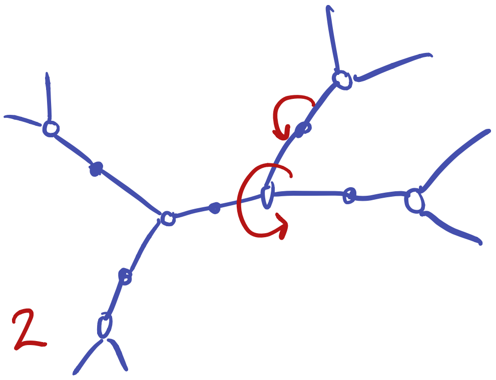
Announcements Mar 9

- Cameras on
- Midterm due Thu 3/11 3:30
- First draft due ~~Mar 26~~ Apr 2
- Office Hours Wed 11-12, Thu 10-10:50, appt

  (no OH Fri this week)



$\begin{pmatrix}1&2\\0&1\end{pmatrix} \in SL_2(\mathbb{Z})[2] = \{A \in SL_2\mathbb{Z} : A \equiv I \bmod 2\} = \ker(SL_2\mathbb{Z} \to SL_2(\mathbb{Z}/2))$

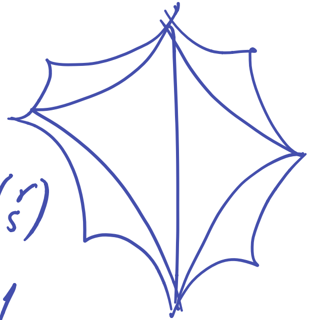
Today: Word problem

Normal forms

BS(1,2)

$\pm\begin{pmatrix}p\\q\end{pmatrix}$ — $\pm\begin{pmatrix}r\\s\end{pmatrix}$

$\left|\det\begin{pmatrix}p&r\\q&s\end{pmatrix}\right| = 1.$



5 Word problem

Given $G = \langle S \mid R \rangle$

$\{S \cup S^{-1}\}^* =$ words in $S \cup S^{-1}$

$\pi : \{S \cup S^{-1}\}^* \longrightarrow G$

Word Problem (Dehn): Determine if a given $w \in \{S \cup S^{-1}\}^*$ has $\pi(w) = id$.

We say WP is solvable if there is an algorithm....

Ball of radius n in $\Gamma_{G,S}$: union of paths from id of length $\leq n$

Equivalent to WP:

① Equality problem (does $\pi(w_1) = \pi(w_2)$) (same as: $\pi(w_1 w_2^{-1}) = id$?)

② determine which paths in Cayley graph are loops.

②' $\exists$ algorithm to draw ball of radius n in the Cayley graph.

First example: $G = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2$

solution to WP: exponent sum.

Second example: $G = \langle a, b \mid \rangle = F_2$

Solution to WP: freely red.

BS(m,n) harder... (later today)

A simple example of a group with unsolvable word problem

Donald J. Collins

Generators:

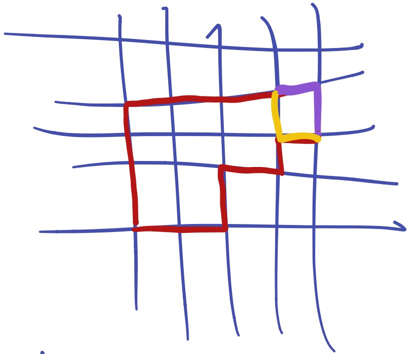
$$a, b, c, d, e, p, q, r, t, k.$$

Relations:

$$p^{10}a = ap,\ p^{10}b = bp,\ p^{10}c = cp,\ p^{10}d = dp,\ p^{10}e = ep,$$
$$qa = aq^{10},\ qb = bq^{10},\ qc = cq^{10},\ qd = dq^{10},\ qe = eq^{10},$$
$$ra = ar,\ rb = br,\ rc = cr,\ rd = dr,\ re = er,$$
$$pacqr = rpcaq, \qquad p^2adq^2r = rp^2daq^2,$$
$$p^3bcq^3r = rp^3cbq^3, \qquad p^4bdq^4r = rp^4dbq^4,$$
$$p^5ceq^5r = rp^5ecaq^5, \qquad p^6deq^6r = rp^6edbq^6,$$
$$p^7cdcq^7r = p^7cdceq^7,$$
$$p^8caaaq^8r = rp^8aaaq^8,$$
$$p^9daaaq^9r = rp^9aaaq^9,$$
$$pt = tp,\ qt = tq,$$
$$k(aaa)^{-1}t(aaa) = k(aaa)^{-1}t(aaa)$$

How can WP be hard?

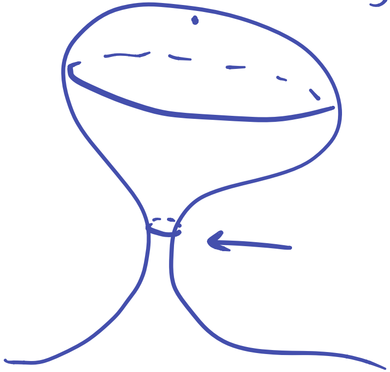
$\mathbb{Z}^2$ example



relations: pushing across squares.

Given a word w with $\pi(w) = id$, can make it monotonically shorter using relations.

To have unsolvable WP must be that short words need <u>many</u> relations (which make the word much longer before getting shorter).



Dehn functions (OHGGT)

Word Problem for BS(1,2)

$BS(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$

Let $G = \{$ linear fns $g: \mathbb{R} \to \mathbb{R}$ of form $g(x) = 2^n x + \alpha$ with $\alpha \in \mathbb{Z}[1/2] \}$

Check: G is a group.

Have $f: BS(1,2) \to G$

$$a \longmapsto g(x) = x + 1$$
$$t \longmapsto g(x) = 2x.$$

Prop. f is an isomorphism.

Cor. f has solvable WP (evaluate $f(w)$).

Pf. Last time: well-def. $f(tat^{-1}) = f(a^2)$

Surj. $f(t^{-k} a^m t^k) = \left(g(x) = x + m/2^k\right)$

$f(a^n) = \left(g(x) = 2^n x\right)$

Inj. Say $f(w) = \mathrm{id}$.

Key: exponent sum on t's is 0.

(take derivative, chain rule)

So: if there are t's there are t^{-1}'s.

Can conjugate so have

$$t \underline{a^k} t^{-1}$$

Replace with a^{2k}.

Eventually $a^n \Rightarrow n = 0$ $\square$

Example

$$\underbrace{t a^2 t^{-1}}_{a^4}\, a\, \underbrace{t^{-1} a^2 t}_{a}\, a\, \underbrace{t^{-1} a t}_{\text{uh oh!}}$$

$$a^4 a a a t^{-1} a t$$

conj by t

$$t a^7 t^{-1} a$$

$$a^{14} a$$

$$a^{15}$$

This shows:

If exp. sum on t is 0

then $w \overset{\text{conj}}{\sim} a^n$

Cayley graph for BS(1,2)

Poll: shortest path to a^{33}