**ANNOUNCEMENTS** Mar 9

- Cameras on
- Midterm due Thu 3/11 3:30
- First draft due **Mar 26** Apr 2
- Office Hours Wed 11-12, Thu 10-10:50, appt (no OH Fri this week)

$(\frac{1}{b^2}) \in \text{SL}_2(\mathbb{F}_2) \Rightarrow \{ A \in \text{SL}_2 : A \equiv I \text{ mod } 2 \}$

$\ker (\text{SL}_2 \to \text{SL}_2(\mathbb{F}_2))$ Today: Word problem Normal forms $\text{BS}(1,2)$

$\phi \mapsto \psi$

$\det (\phi(\psi)) = 1$. 
5 Word problem

Given $G = \langle S \mid R \rangle$

$\{S S' \}^* =$ words in $S S'^{-1}$

$\pi : \{S S' \}^* \rightarrow G$

Word Problem (Dehn): Determine if a given $w \in \{S S' \}^*$ has $\pi(w) = \text{id}$. We say WP is solvable if there is an algorithm...

Ball of radius $n$ in $G_s$: Union of paths from id of length $\leq n$

Equivalent to WP:

1. Equality problem (does $\pi(w_1) = \pi(w_2)$)
   (same as: $\pi(w, w_1) = \text{id}$?)

2. Determine which paths in Cayley graph are loops.

2' Find algorithm to draw ball of radius $n$ in the Cayley graph.

First example: $G = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2$

Solution to WP: exponent sum.

Second example: $G = \langle a, b \rangle = F_2$

Solution to WP: freely red.

BS$(m,n)$ harder... (later today)
A simple example of a group with unsolvable word problem

Donald J. Collins

Generators:

\[ a, b, c, d, e, p, q, r, t, k. \]

Relations:

\[ p^{10}a = ap, \quad p^{10}b = bp, \quad p^{10}c = cp, \quad p^{10}d = dp, \quad p^{10}e = ep, \]
\[ qa = aq^{10}, \quad qb = bq^{10}, \quad qc = cq^{10}, \quad qd = dq^{10}, \quad qe = eq^{10}, \]
\[ ra = ar, \quad rb = br, \quad rc = cr, \quad rd = dr, \quad re = er, \]
\[ pacq = rpcaq, \quad p^{2}adq^{2}r = rp^{2}daq^{2}, \]
\[ p^{3}bcq^{3}r = rp^{3}cbq^{3}, \quad p^{4}bdq^{4}r = rp^{4}dbq^{4}, \]
\[ p^{5}ceq^{5}r = rp^{5}ecaq^{5}, \quad p^{6}deq^{6}r = rp^{6}edboq^{6}, \]
\[ p^{7}cdeoq^{7}r = p^{7}cdeoq^{7}, \]
\[ p^{8}caaq^{8}r = rp^{8}caaq^{8}, \]
\[ p^{9}daaq^{9}r = rp^{9}daaq^{9}, \]
\[ pt = tp, \quad qt = tq, \]
\[ k(aaa)^{-1}t(aaa) = k(aaa)^{-1}t(aaa) \]
How can WP be hard?

$\mathbb{Z}^2$ example

Relations: pushing across squares.

Given a word $w$ with $\Pi(w) = \text{id}$, can make it monotonically shorter using relations.

To have unsolvable WP must be that short words need many relations (which make the word much longer before getting shorter).

Dehn functions (ONGGT)
Word Problem for $BS(1,2)$

$BS(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$

Let $G = \{ \text{linear fns } g : \mathbb{R} \to \mathbb{R} \}$

of form $g(x) = 2^n x + \alpha$ with $\alpha \in \mathbb{Z}[\frac{1}{2}]$

Check: $G$ is a group.

Have $f : BS(1,2) \to G$

$\alpha \mapsto g(x) = x + 1$

$\ t \mapsto g(x) = 2x.$

Prop. $f$ is an isomorphism.

Cor. $f$ has solvable WP (evaluate $f(w)$).

Pf. Last time: well-def. $f(ta^k t^{-1}) = f(a^2)$

Suri. $f(t^{-k} a^m t^k) = (g(w) = x + \frac{m}{2})$

$f(a^n) = (g(x) = 2^n x)$

Inj. Say $f(w) = \text{id}$.

key: exponent sum on $t$'s is 0.

(take derivative, chain rule)

So: if there are $t$'s, there are $t^{-1}$'s.

Can conjugate so have $t a^k t^{-1}$.

Replace with $a^{2k}$.

Eventually $a^n \Rightarrow n=0 \quad \Box$
Example

\[ t^2 t^{-1} a t^{-1} a^2 t a t^{-1} t a \]

\[ a^4 a \]

\[ a a a a t^{-1} t a \]

\[ \text{conj by } t \]

\[ t a t^{-1} a \]

\[ a^{14} a \]

\[ a^{15} a \]

This shows:

If exp. sum on t is 0

then \( w \sim a^n \)
Cayley graph for $BS(1,2)$

Poll: shortest path to $a^{33}$

Hyperbolic space.

Two paths of length 11.