ANNOUNCEMENTS MAR 11

- Cameras on
- Midterm due 11:59 pm
- No HW this week
- First draft due Apr 2
- Office hours by appt.

Today

Normal forms...
in $BS(1,2)$
in B_3
Hyperbolic plane?

Normal Forms

G = group

S = gen set

We have

$$\pi : \{\text{words in } S \cup S^{-1}\} \longrightarrow G$$

A <u>normal form</u> for G is an

$$\eta : G \longrightarrow \{\text{words in } S \cup S^{-1}\}$$

s.t. $\pi \circ \eta = \mathrm{id}$.

To tell if two elements are same, put them in normal form & compare.

This solves word problem.

We can also think of a normal form as a subset of $\{\text{words in } S \cup S^{-1}\}$, one word in $\pi^{-1}(g)$ for each $g \in G$.

<u>Examples</u>. ① $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$

normal form: $\{a^m b^n : m, n \in \mathbb{Z}\}$

② F_2 normal form: freely red. words

Normal forms for $BS(1,2) = \langle a,t \mid tat^{-1} = a^2 \rangle$

We know

$$BS(1,2) \xrightarrow{\cong} \left\{ g(x) = 2^n x + m/2^k : \begin{matrix} m,n,k \\ \in \mathbb{Z} \end{matrix} \right\}$$

$$a \longmapsto g(x) = x+1$$

$$t \longmapsto g(x) = 2x$$

We can check:

$$t^{-k} a^m t^k t^n \longmapsto g(x) = 2^n x + m/2^k.$$

Guess for normal form:

$$\{ t^{-k} a^m t^{k+n} \}$$

FAILS!

$$tat^{-1} = a^2$$

$k=-1$ $m=1$ $n=0$ $\quad$ $k=n=0$, $m=2$

The ambiguity is that $m/2^k$ might be reduced, i.e. m even.

The fix: write elements of $BS(1,2)$ as

$$g(x) = 2^n x + \frac{2m+1}{2^k}$$

or $g(x) = 2^n x$

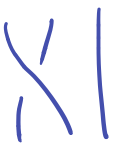
$\rightsquigarrow$ Normal form:

$$\left\{ t^{-k} a^{2m+1} t^{k+n} : \begin{matrix} k,m,n \\ \in \mathbb{Z} \end{matrix} \right\}$$

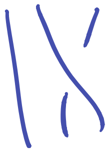
$$\cup \{ t^n : n \in \mathbb{Z} \}$$

Normal form for B_3 (or B_n)

Generators:

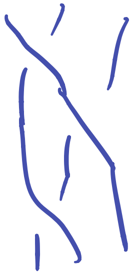


σ_1



σ_2

Mutiplication is stacking.



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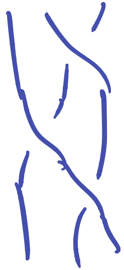


$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 = \Delta$$

Poll. Which are equiv to

1 2 1 1 2 2 2 ?

2 1 2 1 2 2 2

2 1 1 2 1 2 2

2 1 1 1 2 1 2

2 1 1 1 1 2 1

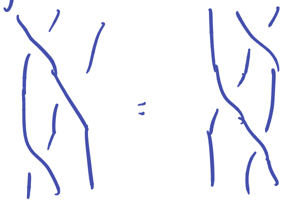
Garside Normal Form

Ingredient # 1 :

$$B_3 \longrightarrow \mathbb{Z}$$
$$\sigma_1 \longmapsto 1$$
$$\sigma_2 \longmapsto 1$$

"signed word length"

Ingredient # 2 :



"half-twist"

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 = \Delta$$

Running example: $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}$

Step 1. Replace each σ_i^{-1} with $\Delta^{-1}\cdot$ pos. word.

Why can we do this?

$$\Delta^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} \quad \text{and} \quad \Delta^{-1} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$$

$$\Delta^{-1}\sigma_2\sigma_1 = \sigma_2^{-1} \qquad \Delta^{-1}\sigma_1\sigma_2 = \sigma_1^{-1}$$

example. $\sigma_1\sigma_2\sigma_1\underline{\sigma_2^{-1}} \rightsquigarrow \sigma_1\sigma_2\sigma_1\underline{\Delta^{-1}\sigma_2\sigma_1}$

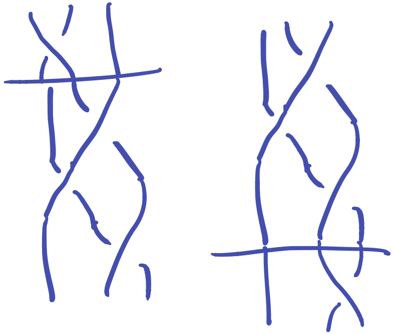
Step 2. Move all Δ^{-1} to the left.

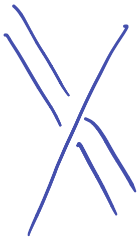
Why can we do this? $\sigma_i\Delta^{-1} = \Delta^{-1}\sigma_{n-i}$

example. $\rightsquigarrow \Delta^{-1}\sigma_2\sigma_1\sigma_2\sigma_2\sigma_1$

Check:

$\sigma_i \Delta^{-1} = \Delta^{-1} \sigma_{n-i}$





We now have $\Delta^i \cdot$ pos word $i \leq 0$.

<u>Step 3</u>. Find maximal i so our braid is

$\Delta^i \cdot$ pos word $i \leq 0$.

In our example:

$$\Delta^{-1} \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1 = \cancel{\Delta^{-2} \overbrace{\sigma_2 \sigma_1 \sigma_2}^{\Delta} \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1}$$

How do we know our example is not

$\Delta^0 \cdot$ pos word ?

It is!

$\Delta^0 \sigma_2 \sigma_1$

In general, use ingredient #1.

Alternate example: (not related to running example)

How do I know

$$\underset{\substack{\downarrow \\ \text{signed word} \\ \text{length } -2}}{\Delta^{-1}\sigma_1} \neq \Delta^0 \cdot \underset{\substack{\downarrow \\ \text{signed word length} \\ \geq 0.}}{\text{pos word}} \; ?$$

Another example:

How do I know

$$\underset{\text{signed word length } 1}{\Delta^{-1}\sigma_1^4} \neq \Delta^0 \cdot \underbrace{\text{pos word}}_{\text{must have length } 1} \; ? \quad \text{only 2 such words.}$$

Step 4 Find all $\Delta^i \cdot$ pos word equalling g.

Choose the smallest in lexicographic order.

example. $\Delta^0 \sigma_2 \sigma_1 \longleftarrow$ normal form.

only candidates are $\cancel{\Delta^0 \sigma_1 \sigma_2}$

$\cancel{\Delta^0 \sigma_1^2}$

$\cancel{\Delta^0 \sigma_2^2}$

Steps 3 & 4 use:

Thm. If two positive braids are equal, they differ by finitely many

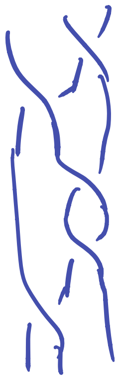
$121 \longleftrightarrow 212$

no inverses needed!

In fancy language:

The braid monoid B_n^+ embeds into B_n.

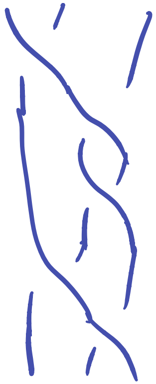
Prove this theorem?



21221

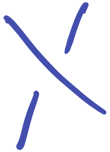
12121

11211



12212

positive crossing



neg. crossing