

Announcements Mar 11

- Cameras on
- Midterm due 11:59 pm
- No HW this week
- First draft due Apr 2
- Office hours by appt.

Today

Normal forms...
in BS(1,2)
in B₃
Hyperbolic plane?
Normal Forms

G = group
S = gen set

We have

\( \pi : \{ \text{words in } S \cup S^{-1} \} \rightarrow G \)

A normal form for G is an

\( \eta : G \rightarrow \{ \text{words in } S \cup S^{-1} \} \)

s.t. \( \pi \circ \eta = \text{id} \).

To tell if two elements are same, put them in normal form & compare.

This solves word problem.

We can also think of a normal form as a subset of \( \{ \text{words in } S \cup S^{-1} \} \), one word in \( \pi^{-1}(g) \) for each \( g \in G \).

**Examples.**

1. \( \mathbb{Z}^2 = \langle a, b \mid ab=ba \rangle \)
   
   normal form: \( \{a^m b^n : m, n \in \mathbb{Z} \} \)

2. \( \mathbb{F}_2 \) normal form: freely reduced words
Normal forms for $BS(1,2) = \langle a, t \mid tat^{-1} = a^2 \rangle$

We know

$$BS(1,2) \xrightarrow{\cong} \{ g(x) = 2^n x + \frac{m}{2^k} : m, n, k \in \mathbb{Z} \}$$

- $a \mapsto g(x) = x + 1$
- $t \mapsto g(x) = 2x$

We can check:

$$t^{-k} a^m t^k t^n \mapsto g(x) = 2^n x + \frac{m}{2^k}.$$ 

Guess for normal form:

$$\{ t^{-k} a^m t^{k+n} \}$$

FAILS!

$$t a t^{-1} = \begin{cases} 2 & k = -1, m = 1, n = 0, k = n = 0, m = 2 \end{cases} \cup \{ t^n : n \in \mathbb{Z} \}$$

The ambiguity is that $m/2^k$ might be reduced, i.e. $m$ even.

The fix: write elements of $BS(1,2)$ as

$$g(x) = 2^n x + \frac{2m+1}{2^k}$$

or $g(x) = 2^n x$.

Normal form:

$$\{ t^{-k} 2^{m+n} t^{k+n} : k, m, n \in \mathbb{Z} \}$$
Normal form for $B_3$ (or $B_n$)

Generators:

\[ \sigma_1, \quad \sigma_2 \]

Multiplication is stacking.

\[ \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 = \Delta \]

Poll: Which are equiv to

\[ \begin{align*}
1 & \ 2 & \ 1 & \ 1 & \ 2 & \ 2 & \ 2 \\
2 & \ 1 & \ 2 & \ 1 & \ 2 & \ 2 & \ 2 \\
2 & \ 1 & \ 1 & \ 2 & \ 1 & \ 2 & \ 2 \\
2 & \ 1 & \ 1 & \ 1 & \ 2 & \ 1 & \ 2 \\
2 & \ 1 & \ 1 & \ 1 & \ 1 & \ 2 & \ 1 \\
\end{align*} \]
Garside Normal Form

Ingredient # 1 :

\[ B_3 \rightarrow \mathbb{Z} \]
\[ \sigma_1 \mapsto 1 \]
\[ \sigma_2 \mapsto 1 \]

“signed word length”

Ingredient # 2 :

\[ \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 = \Delta \]

“half-twist”

Running example : \( \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \)

Step 1. Replace each \( \sigma_i^{-1} \) with \( \Delta^{-1} \) pos. word.

Why can we do this?

\[ \Delta = \sigma_2 \sigma_1 \sigma_2 \]
\[ \Delta^{-1} \sigma_2 \sigma_1 = \sigma_2^{-1} \]
\[ \Delta^{-1} \sigma_1 \sigma_2 = \sigma_1^{-1} \]

example. \( \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sim \sigma_1 \sigma_2 \sigma_1 \Delta^{-1} \sigma_2 \sigma_1 \]

Step 2. Move all \( \Delta^{-1} \) to the left.

Why can we do this? \( \sigma_i \Delta^{-1} = \Delta^{-1} \sigma_n \)

example. \( \sim \Delta^{-1} \sigma_2 \sigma_1 \sigma_2 \sigma_2 \sigma_1 \)
We now have $\Delta^i \cdot \text{pos word } i \leq 0$.

**Step 3.** Find maximal $i$ so our braid is $\Delta^i \cdot \text{pos word } i \leq 0$.

In our example:

$\Delta^i \sigma_2 \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1$.

How do we know our example is not $\Delta^0 \cdot \text{pos word }$?

It is! $\Delta^0 \sigma_2 \sigma_1$.

In general, use ingredient #1.
Alternate example: (not related to running example)

How do I know \( \Delta \tilde{\sigma}_1 \neq \Delta^0 \cdot \text{pos word?} \)

\[ \downarrow \quad \downarrow \]

signed word length -2

signed word length \( \geq 0 \).

Another example:

How do I know \( \Delta \tilde{\sigma}_4 \neq \Delta^0 \cdot \text{pos word?} \)

signed word length 1

\( \text{must have length} \quad \frac{1}{1} \) only 2 such words.
Step 4: Find all $\Delta^i$ pos word equaling $g$.
Choose the smallest in lexicographic order.

**Example.** $\Delta^0 \sigma_2 \sigma_1 \leftarrow$ normal form.
Only candidates are $\Delta^0 \sigma_1 \sigma_2$, $\Delta^0 \sigma_2 \sigma_1$, $\Delta^0 \sigma_2$, $\Delta^0 \sigma_1$, $\Delta^0$.

Steps 3 & 4 use:

**Thm.** If two positive braids are equal, they differ by finitely many.

$121 \leftrightarrow 212$

no inverses needed!

In fancy language:
The braid monoid $B^n_t$ embeds into $B^n$. 
Prove this theorem?

21221
12121
12111

12212

positive crossing

neg. crossing