ANNOUNCEMENTS MAR 18

- Cameras on
- First draft due Apr 2
- HW due Thu 3:30
- Office Hours Fri 2-3, Tue 11-12, appt
- Talk to me about extra credit.

<u>Question</u>. What are all groups G of order $n!$ with $[G,G] = A_n$?

Hope: $G = S_n$.

<u>Today</u>

Burnside problem

$$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\rangle$$

"lantern relation"

Burnside Problem

A group is a <u>torsion group</u> if all elements have finite order.

Finite groups are all torsion gps

Easy to make infinite torsion groups:

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \cdots$$

$$\mathbb{Q}/\mathbb{Z}$$

These are not finitely generated (why?)

Q. (Burnside 1902) Is there a fin. gen. infinite torsion group?

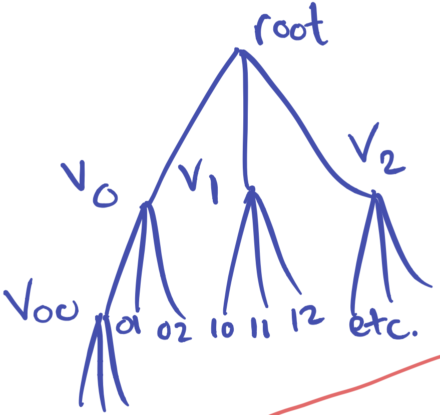
A. (Golod-Shafarevich '60s)

Yes.

We'll show an example from 80's by Gupta-Sidki using GGT.

Starting pt is...

T = rooted ternary tree



uncountable

$\mathrm{Sym}(T)$ = root-preserving symmetries.

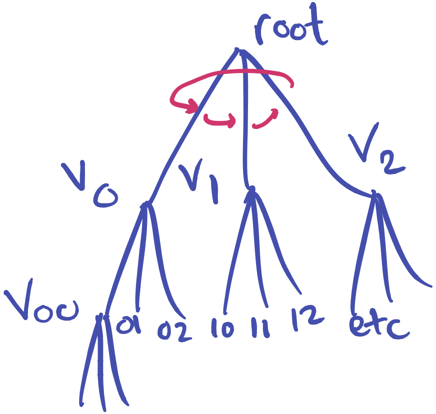
Important thing:

$$\mathrm{Sym}(T) \times \mathrm{Sym}(T) \times \mathrm{Sym}(T) \leq \mathrm{Sym}(T)$$

$\mathrm{Sym}(T)$ is self-similar.

Two elements of Sym(T)

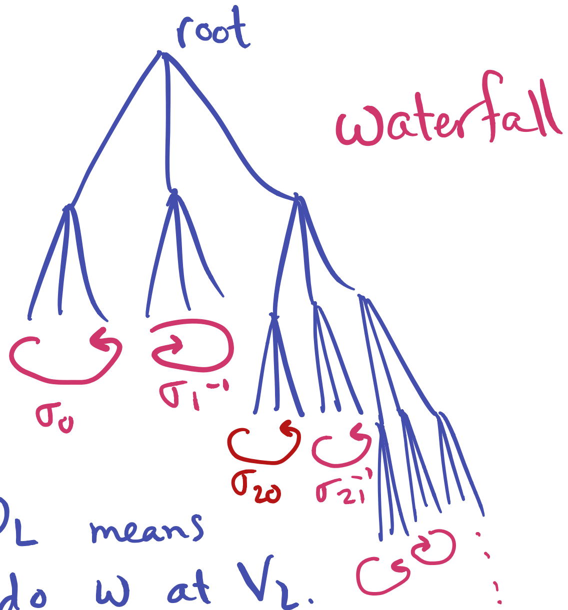
$\sigma : V_{n_1 \cdots n_k} \to V_{(n_1+1) n_2 \cdots n_k}$ mod 3



σ_L means do σ at V_L.

e.g. σ_0

$\omega = \sigma_0 \sigma_1^{-1} \sigma_{20} \sigma_{21}^{-1} \sigma_{220} \sigma_{221}^{-1} \cdots$



ω_L means do ω at V_L.

Let $U = \langle \sigma, \omega \rangle \leq \mathrm{Sym}(T)$

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Thm. U is a fin. gen. ∞ torsion group.

⓪ fin gen ✓

① ∞

② torsion

Need a "normal form" for U.

Then will do ① & ②

A "normal form" for U

<u>Lemma 1</u>. Each elt of U can be expressed as

$$\sigma^k x_1 \cdots x_n$$

where $x_i \in \{\omega, \sigma\omega\sigma^{-1}, \sigma^2\omega\sigma^{-2}\}$

(kind of like B_n normal form).

Pf. Need a relation

$\sigma = \sigma^{-2} \Rightarrow \omega\sigma = \omega\sigma^{-2}$

$$\Rightarrow \boxed{\omega\sigma = \sigma(\sigma^2\omega\sigma^{-2})}$$

Use the relation to push σ's to the left.

<u>example</u>

$$\omega\sigma\underline{\sigma\omega}$$

$$\underbrace{\sigma(\sigma^2\omega\sigma^{-2})}_{x_1}\ \sigma\ \underbrace{\omega}_{x_2}$$

bad

$$\sigma^2\ \sigma^2(\sigma^2\omega\sigma^{-2})\ \sigma^{-2}\ \omega$$

$$\sigma^2 \underbrace{(\sigma\omega\sigma^{-1})}_{x_1}\underbrace{(\omega)}_{x_2} \qquad \square$$

① **Prop.** $|U| = \infty$.

We will find $K \lneq U$

and $K \twoheadrightarrow U$.

The Prop follows.

Defining K

have $U \twoheadrightarrow \mathbb{Z}/3$

action on three edges from root. Works because ω & σ preserves the cyclic order.

K is the kernel.

(*) In terms of "normal form" these are the $\sigma^k x_1 \cdots x_n$ with $k=0$.

Let $H = \langle \omega, \sigma\omega\sigma^{-1}, \sigma^2\omega\sigma^{-2} \rangle$

Lemma. $K = H$.

Pf. Step 1. $H \leq K$ ✓

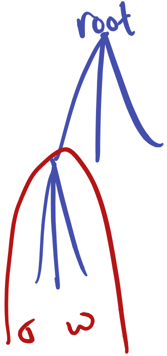
Step 2. $H \trianglelefteq U$.

finite check. → conjugate each gen for H by gen for U, end up back in H

Step 3. $U/H \cong \mathbb{Z}/3$ by Lemma 1

Lemma $K \twoheadrightarrow U$

Pf. K maps to the copy of $\mathrm{Sym}(T)$ below vertex 0.



Want: Image of K contains σ_0 & ω_0

Check on generators:

$$\omega \longmapsto \sigma_0$$
$$\sigma\omega\sigma^{-1} \longmapsto \omega_0$$
$$\sigma^2\omega\sigma^{-2} \longmapsto \sigma_0^{-1}$$

$\square$

② **Prop.** U is torsion: each elt has order a power of 3.

Pf. Induction on syllable length in normal form.

$$\sigma^k x_1 \cdots x_n$$

σ^k is a syllable, x_i is a syllable.

Idea. Given g, show g^3 is a product of 3 commuting elements of shorter syllable length

Proof by example

$g = \sigma\omega$ syll. length 2.

$\rightsquigarrow g^3 = \sigma\omega\sigma\omega\sigma\omega$

normal form has $k=0$:

$\cancel{\sigma^3}(\sigma^{-2}\omega\sigma^2)(\sigma^{-1}\omega\sigma)\omega$

lie in 3 diff. factors

$\mathrm{Sym}(T) \times \mathrm{Sym}(T) \times \mathrm{Sym}(T)$

all three pieces have syllable length 1.

$\square$