Chapter 2: Morphisms

Polynomial Maps

Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ aff. alg. vars.

Defn. $f: X \to Y$ is a morphism if it is the restriction of a polynomial map.
That is, $\exists\ f_1, \dots, f_m \in k[x_1, \dots, x_n]$
s.t. $f(x) = (f_1(x), \dots, f_m(x))\ \forall\ x \in X$.

Fact. Morphisms are continuous in the Zariski topology.

Pf. $f^{-1}(Z(h_1, \dots, h_r)) = Z(h_1 \circ f, \dots, h_r \circ f)$.

Def. A morphism is an isomorphism if it has an inverse.

Example. An affine change of coords on $\mathbb{A}^n$ is a morphism:

$$\begin{aligned} \mathbb{A}^n &\to \mathbb{A}^n \\ x &\mapsto (L_1(x), \dots, L_n(x)) \end{aligned}$$

$$L_i(x) = \lambda_{i1} x_1 + \dots + \lambda_{in} x_n + \mu_i$$

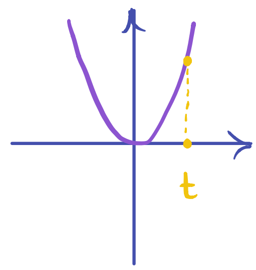
This is invertible iff (λ_{ij}) is.

Example. $\mathbb{A}^2 \to \mathbb{A}^1$

$(x,y) \longmapsto x$

is a morphism. It is not an isomorphism since it is not invertible.

Example. $X = Z(y-x^2) \subseteq \mathbb{A}^2$



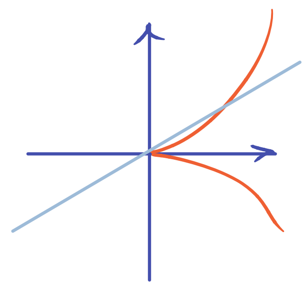
$\mathbb{A}^1 \longrightarrow X$

$t \longmapsto (t,t^2)$

is an isomorphism, with inverse

$(t,t^2) \longmapsto t$

Example. $X = Z(y^2-x^3) \subseteq \mathbb{A}^2$



bijection given by: the line of slope t intersects X at (t^2,t^3)

$f: \mathbb{A}^1 \longrightarrow X$

$t \longmapsto (t^2,t^3)$

is bijective, but not an isomorphism.

One candidate inverse is $(x,y) \longmapsto y/x$.

How do we know this is not a polynomial?

We need a new tool for this!

The projection $\mathbb{A}^2 \to \mathbb{A}^1$ shows that morphisms do not always map varieties to varieties. The hyperbola $X = Z(xy-1) = \{(t,t^{-1}) : t \neq 0\}$ is a closed set and the image is $\mathbb{A}^1 \setminus \{0\}$, which is not closed.

The Coordinate Ring

Let $X \subseteq \mathbb{A}^n$ be an aff. alg. var., $f \in k[x_1, \dots, x_n]$

$\rightsquigarrow$ restriction $f|_X$

The coordinate ring of X is

$$k[X] = \{ f|_X : f \in k[x_1, \dots, x_n] \}$$

$$= \{ \text{polynomial fns on } X \}$$

$k[X]$ is a ring, in fact a k-algebra. In fact:

$$k[X] \cong k[x_1, \dots, x_n] / I(X).$$

Example. Let $X = Z(xy - 1)$. Then $1/x$ lies in $k[X]$ since it is equivalent to the polynomial y.

Note. $k[\mathbb{A}^n] = k[x_1, \dots, x_n]$

$k[p] \cong k$ (see the above proof that $(x_1 - a_1, \dots, x_n - a_n)$ is maximal)

$k[p_1, \dots, p_r] \cong k^r$

Example. $X = Z(x^2+y^2-z^2) \subseteq \mathbb{A}^3$ cone $\quad k=\mathbb{C}$

$x^2+y^2-z^2$ irreducible (if not, it would be a product of 2 linear, homogeneous factors...)

See also Stackexchange #486668

$\Rightarrow (x^2+y^2-z^2)$ prime, hence radical

$\Rightarrow I(X) = (x^2+y^2-z^2) \quad$ (SN)

$\Rightarrow \mathbb{C}[X] = \mathbb{C}[x,y,z]/(x^2+y^2-z^2)$

Can say: $\mathbb{C}[x,y,z]$ equipped with the relation

$$x^2+y^2-z^2=0.$$

So: $x^3+2xy^2-2xz^2+x$
$$= 2x(x^2+y^2-z^2)+x-x^3$$
$$= x-x^3$$

Example. $X = Z(y-x^2)$

Every $f \in k[X]$ can be written as a poly. in x (just replace all y's with x^2).

Prop. X irred $\Longleftrightarrow k[X]$ an integral domain.

no 0-divisors

Pf. R/J an int. dom. $\Longleftrightarrow J$ prime.

Fact. $k[X]$ is the (fin. gen.) k-alg generated by the coordinate functions $X \to k$, $x \mapsto x_i$.

Example. $X = Z(y-x^2, z-x^3)$ "twisted cubic"

Claim: $I(X) = (y-x^2, z-x^3)$

think of f as poly in y, divide by the linear poly $y-x^2$ to get g, etc.

Pf: Let $f \in I(X)$. Use div alg wrt y, then z

$\rightsquigarrow f(x,y,z) = (y-x^2)g(x,y,z) + (z-x^3)h(x,z) + r(x)$

Then $\forall\, t \in k$, $(t,t^2,t^3) \in X$, so $r(t)=0\ \forall t$

$\Rightarrow r=0$, whence the claim.

In $k[X]$: $y=x^2$, $z=x^3 \rightsquigarrow k[X] \cong k[x]$, an integral domain. $\Rightarrow X$ irred.

Another proof: $\mathbb{A}^1 \to X$ $\quad t \mapsto (t,t^2,t^3)$ is a surjective morphism $\Rightarrow X$ irred.

Dictionary

As with all of $\mathbb{A}^n$, there is a dictionary:

$$\begin{array}{ccc} \text{subvarieties} & \longleftrightarrow & \text{radical ideals} \\ Y \subseteq X & & J \subseteq k[X] \\ \text{irred subvarieties} & \longleftrightarrow & \text{prime ideals} \\ \text{pts} & \longleftrightarrow & \text{max ideals} \end{array}$$

Pf. Homework! $\quad$ 3rd isom. thm!

Pullback

A morphism $f: X \to Y$ induces a pullback

$$f_*: k[Y] \to k[X]$$

$$g + I(Y) \longmapsto g \circ f + I(X)$$

This is well def. since the composition of polynomials is a polynomial, and because if $g \in I(Y)$ then $g \circ f$ lies in $I(X)$.

Note:
- f_* is a k-algebra homom.
- $(fg)_* = g_* f_*$
- f an isomorphism $\Rightarrow$ f_* is

Example.
$$\mathbb{A}^1 \longrightarrow Z(y - x^2) \subseteq \mathbb{A}^2$$
$$t \longmapsto (t, t^2)$$
(already said this was $\cong$)

Pullback:

$$\mathbb{C}[x,y]/(y-x^2) \longrightarrow \mathbb{C}[t]$$
$$x \longmapsto t$$
$$y \longmapsto t^2$$

surjective with trivial kernel, hence $\cong$.

Example. $\mathbb{A}^1 \longrightarrow Z(y^2-x^3) \subseteq \mathbb{A}^2$

$t \longmapsto (t^2, t^3)$

Pullback:

$$\mathbb{C}[x,y]/(y^2-x^3) \longrightarrow \mathbb{C}[t]$$
$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

Not an $\cong$ since t not in the image.

So the map $\mathbb{A}^1 \longrightarrow X$ is not $\cong$.

Example. Is $X = Z(xy-1)$ isomorphic to $\mathbb{A}^1$?

No.

Have: $k(\mathbb{A}^1) = k[x]$

$k(X) = k[x, x^{-1}]$ Laurent polynomials

Want to show these are not isomorphic.

Suppose $\Phi : k[x,x^{-1}] \to k[x]$ is an isomorphism.

$\Longrightarrow \Phi(1) = 1$

$\Longrightarrow \Phi(x)\Phi(x^{-1}) = 1$

$\Longrightarrow \Phi(x), \Phi(x^{-1})$ units in $k[x]$

$\Longrightarrow$ they are scalars

$\Longrightarrow \operatorname{Im}\Phi \subseteq$ scalars. CONTRADICTION.

More Geometry vs. Algebra

An algebra is reduced if it has no nilpotent elts: $r^n = 0$.

k = alg. closed.

Thm (i) Every $k[X]$ is a fin. gen. red. k-alg.
(ii) Every finitely gen. reduced k-algebra is isom. to some $k[X]$.
(iii) If $f : X \to Y$ is a morphism, then
$$f_* : k[Y] \to k[X]$$
is a homomorphism.
(iv) If $\sigma : R \to S$ is a homom. of reduced fin. gen. k-algebras, then there is a morphism $f : X \to Y$ with $f_* = \sigma$. This f is unique up to isomorphism.

In other words the categories of aff. alg. var's & fin. gen. red. k-alg's are (anti-) isomorphic.

Note. In 1950's Grothendieck removed 3 hypotheses: fin gen, red, alg closed. The corresponding geometric objects are affine schemes.

Pf. (i) $k[X]$ is gen. by. images of the x_i.
$I(X)$ radical $\Rightarrow$ reduced.

(ii) Let R be a fin. gen. red. k-alg.
Choose a "presentation". If the gens are $y_1, \dots, y_m$,
then $R \cong k[y_1, \dots, y_m]/J$ ← relations

(J is kernel of $k[y_1, \dots, y_m] \to R$)
R reduced $\Rightarrow$ J radical
Let $Y = Z(J)$. SN $\Rightarrow k[Y] \cong R$.

(iii) We already know this.

(iv) Fix $\sigma : R \to S$. As above:
$$\sigma : k[y_1, \dots, y_m]/J \longrightarrow k[x_1, \dots, x_n]/I$$
Again, I & J radical.
$\Rightarrow R, S$ are coord rings of $Z(J)$ & $Z(I)$.

Want polynomial $f : \mathbb{A}^n \to \mathbb{A}^m$
s.t. $Z(I) \longmapsto Z(J)$
& $f_* = \sigma$

Let $\tilde{\sigma} : k[y_1, \dots, y_m] \longrightarrow S$

$\qquad f \longmapsto \sigma([f])$

This is a k-alg. homom. (it's the composition of 2 such)

Let f_i = rep. of $\tilde{\sigma}(y_i)$

i.e. $\tilde{\sigma}(y_i) = f_i + J$

Define $f : \mathbb{A}^n \longrightarrow \mathbb{A}^m$

$x \longmapsto (f_1(x), \dots, f_n(x))$

Claim 1. $f(Z(I)) \subseteq Z(J)$

Pf. $\forall\, x \in Z(I)$ want $f(x) \in Z(J)$

i.e. $g \circ f(x) = 0 \quad \forall\, g \in J,\, x \in Z(I)$

i.e. $g \circ f \in I$

Fix a $g \in J$. Have:

$$g \circ f + I = g(f_1, \dots, f_n) + I$$

These terms make sense. They are independent of choice of rep of $\tilde{\sigma}(x_i)$.

$$\begin{aligned} &= g(\tilde{\sigma}(x_1), \dots, \tilde{\sigma}(x_n)) + I \\ &= \tilde{\sigma}(g(x_1, \dots, x_n)) + I \\ &= \tilde{\sigma}(g) + I \\ &= 0 + I \quad ✓ \end{aligned}$$

Claim 2. $f_* = \sigma$

Pf. Let $g \in k[y_1, \dots, y_m]$

$$f_*(g+J) = g \circ f + I$$
$$= \tilde{\sigma}(g) + I \quad \text{(as above)}$$
$$= \sigma(g+J) \quad ✓$$

We really should have done this claim first.
The previous claim is the special case $f_*(0) = \sigma(0)$

Claim 3. f is unique: If $f, g : X \to Y$ have $f_* = g_*$ then $f = g$.

Pf. Write $I(Y) = k[y_1, \dots, y_m] / J$.

Have $f^*(y_i + J) = g^*(y_i + J)$

$\rightsquigarrow \ y_i \circ f + I = y_i \circ g + I$

Say $f = (f_1, \dots, f_m)$ $g = (g_1, \dots, g_m)$

$\rightsquigarrow \ f_i + I = g_i + I$

$\rightsquigarrow \ f + I = g + I$

i.e. $f|X = g|X$ ✓

There is a loose end: we didn't show that the X & Y we constructed are unique.

Prop. $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ aff. alg. var's.
$f : X \to Y$ a morphism.
Then f is an isomorphism $\Leftrightarrow$ f^* is.

Pf. ($\Rightarrow$) done already.
($\Leftarrow$) f^* an iso
$\Rightarrow \exists\ \sigma : k[X] \to k[Y]$
s.t. $f^* \circ \sigma = \mathrm{id}$ & $\sigma \circ f^* = \mathrm{id}$.
Any such σ is g^* for some $g : Y \to X$
(We gave the argument above. It's just that instead of starting with random R & S we start with $k[X]$ & $k[Y]$).
Now: $f^* g^* = (gf)^* = \mathrm{id}$
$\Rightarrow gf = \mathrm{id}$ (Claim 3). ✓

Cor. X, Y aff. alg. vars.
Then $X \cong Y \Leftrightarrow k[X] \cong k[Y]$

Pf. ($\Rightarrow$) done.
($\Leftarrow$) Any $k[X] \to k[Y]$ gives $Y \to X$ as above.
Apply the Prop. $\square$

Dictionary (= Functor)

Geometry	Algebra
aff. alg. var.	fin gen red. k-alg R
alg. subset	rad. ideal in R
irred. alg. subset	prime ideal in R
point	max ideal in R
poly. map	k-alg homom.

For a more organized exposition of this last theorem, see Moraru.

Dimension

X = aff. alg. var.

Def $\dim X$ = supremum of lengths of chains $X \supset X_1 \supset \cdots \supset X_d$ of distinct irred. aff. alg. var.'s.

Fact. $\dim X = \max \dim X_i$ where $\{X_i\}$ are the irred. components.

Fact. If $X \subseteq Y$ then $\dim X \leq \dim Y$.

So: $\dim X = 0 \iff X = \text{pt}$.

By the above dictionary:

$\dim X$ = krull dim of $k[X]$.

Some names:

0-dim	pts
1-dim	curve
2-dim	surface
n-dim	n-fold

Problem. What is $\dim \mathbb{A}^n$?

Obviously, $\dim \mathbb{A}^n \geq n$. Will (almost) prove:

Thm. $\dim X = \text{transc.}\deg_k k(X)$ — field of fractions for $k[X]$ = poly. fns. on X.

Def. For a comm. ring A, $x \in A$

$$S_{\{x\}} = \{x^n(1-ax) : n \in \mathbb{N}, a \in A\}$$

[This is a multiplicative set. Check this!]

The boundary $A_{\{x\}}$ of A at x is the ring of fractions $S_{\{x\}}^{-1}A$.

Fact 1. S = mult. subset of a ring A

$$\left\{\begin{array}{c}\text{prime ideals}\\ \text{disjoint from } S\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{prime ideals}\\ \text{in } S^{-1}A\end{array}\right\}$$

$$\mathfrak{p} \longrightarrow S^{-1}\mathfrak{p} = (S^{-1}A)\mathfrak{p}$$

$$\text{inverse image} \longleftarrow \mathfrak{q}$$

Pf. Milne 1.14

Fact 2. A = ring. $\forall x \in A$, max ideal $\mathfrak{m} \subset A$

$$\mathfrak{m} \cap S_{\{x\}} \neq \emptyset.$$

Pf. If $x \in \mathfrak{m}$ then $x = x^1(1 - 0x) \in S_{\{x\}}$

If $x \notin \mathfrak{m}$ then x is invertible mod $\mathfrak{m}$

$\implies \exists\, a$ s.t. $1 - ax \in \mathfrak{m}$ $\square$

Fact 3. A = ring, $\mathfrak{m} \subset A$ max. ideal, $\mathfrak{p} \subseteq \mathfrak{m}$ prime.

$\forall\ x \in \mathfrak{m} \setminus \mathfrak{p}$, $\mathfrak{p} \cap S_{\{x\}} = \emptyset$.

Pf. Suppose not: $x^n(1 - ax) \in \mathfrak{p}$.

$\implies 1 - ax \in \mathfrak{p} \implies 1 - ax \in \mathfrak{m} \implies 1 \in \mathfrak{m}$ $\square$

Recall: Krull dim = max length of a chain of prime ideals.

Prop. A = ring, $n \in \mathbb{N}$.

Krull dim $A \leq n \iff \forall x \in A$, Krull dim $A_{\{x\}} \leq n - 1$

Pf. Fact 1: $\left\{ \begin{array}{c} \text{prime ideals} \\ \text{disjoint from } S_{\{x\}} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{prime ideals} \\ \text{in } A_{\{x\}} \end{array} \right\}$

Fact 2: A chain of prime ideals beginning with a max ideal gets shortened in $A_{\{x\}}$.

Fact 3: Such a chain gets shortened by at most 1. $\square$

Prop. A = integral domain.
$F(A)$ = field of fractions
$k \subseteq A$ subfield.
Then $\text{tr.deg}_k F(A) \geq \text{krull dim } A$

Pf. WLOG k alg. closed.
If tr. deg $= \infty$, nothing to prove.
Say $\text{tr.deg}_k F(A) = n$. Induction.
Let $x \in A$.

- If $x \notin k$, then x transc. over k

$$\Longrightarrow \text{tr.deg}_{k(x)} F(A) = n-1$$

Since $F(A) \cong F(A_{\{x\}})$, have: $\text{tr deg}_{k(x)} F(A_{\{x\}}) = n-1$
Since $k(x) \subseteq A_{\{x\}}$, induction gives

$$\text{krull dim } A_{\{x\}} \leq n-1.$$

Previous Prop $\Longrightarrow$ $\dim A \leq n$.

- If $x \in k$, then $0 = 1 - x^{-1}x \in S_{\{x\}} \Longrightarrow A_{\{x\}} = 0$.

Again $\dim A_{\{x\}} \leq n-1$ $\square$

Cor. $\text{krull dim } k[x_1, \dots, x_n] = n$.
Pf. ($\geq$) $(x_1, \dots, x_n) \supset (x_1, \dots, x_{n-1}) \supset \cdots \supset (x_1) \supset 0$
($\leq$) Prev. Prop. $\square$

Cor. $\dim \mathbb{A}^n = n$.

Hypersurfaces

Prop. A hypersurface in $\mathbb{A}^n$ has dim $n-1$.

Pf. Let H be a hypersurface
WLOG H irred.
$\Rightarrow H = Z(f)$ f irred.
Let $k[x_1,\dots,x_n] = k[X_1,\dots,X_n]/(f)$ $x_i = X_i + (f)$
and $k(x_1,\dots,x_n)$ the field of fractions.
$f \neq 0 \Rightarrow$ some X_i, say X_n, appears in it.
$\Rightarrow$ no nonzero poly in $X_1,\dots,X_{n-1}$ lies in (f).
$\Rightarrow x_1,\dots,x_{n-1}$ alg indep.
But x_n is alg. over $x_1,\dots,x_{n-1}$ (think of f as a poly. in x_n w/ coeffs in $k[x_1,\dots,x_{n-1}] \subseteq k(x_1,\dots,x_n)$)
$\Rightarrow \{x_1,\dots,x_{n-1}\}$ is a transc. basis for $k(x_1,\dots,x_n)$ over k. Apply the theorem. $\square$

Example. Say $f(x,y)$, $g(x,y)$ nonconstant, no common factors.
Then $\dim Z(f) = 1$ by (2)
Also: $\dim Z(f,g) < \dim Z(f)$
$\Rightarrow Z(f,g)$ = finite set of points.
How many? Stay tuned (Bézout).

Prop. The closed sets of codim 1 in $\mathbb{A}^n$ are exactly the hypersurfaces.

Pf. Say W = aff. alg. var of codim 1.

$W_1, \ldots, W_s$ the irred components.

$I(W) = \bigcap I(W_i)$, so if $I(W_i) = Z(f_i)$

then $I(W) = Z(f_1 \cdots f_r)$.

Thus, WLOG W irred.

$I(W)$ is prime, nonzero.

Let f be an irred. poly in $I(W)$.

$\rightsquigarrow (f)$ prime.

If $(f) \neq I(W)$ then

$$I(W) \supset f \supset (0) \quad \text{distinct primes.}$$

$\Longrightarrow \mathbb{A}^n \supset Z(f) \supset W$

$\Longrightarrow \operatorname{codim} W > 1.$ $\square$

Classification of Irred. Aff. Alg. Vars in $\mathbb{A}^2$

dim 2 : $\mathbb{A}^n$ $\longleftrightarrow$ (0)

dim 1 : hypersurfaces $\longleftrightarrow$ (f) f irred.

$V = V(f)$, where f is any irreducible in $I(V)$.

dim 0 : pt. $\longleftrightarrow$ $(x_1 - a_1, x_2 - a_2)$

Noether Normalization

Needed for the theorem...

Say a k-alg. B is finite over a k-alg A if there are $b_1, \dots, b_n$ s.t. A-span of the b_i is B

e.g. $k[x]$ is finitely gen. over k but not finite over k.

Say $b \in B$ is integral over A if

$$b^n + a_{n-1} b^{n-1} + \dots + a_0 = 0.$$

Fact. b integral over $A \iff A[b]$ finite over A

Thm. A = fin gen. k-alg. $\exists\, x_1, \dots, x_d \in A$ alg indep. over k s.t. A is finite over $k[x_1, \dots, x_d]$

Pf (assuming k infinite)

Let $A = k[x_1, \dots, x_n]$ (really, a quotient of this)

Induct on n.

If $\{x_i\}$ alg indep., nothing to prove

Otherwise, you show A is finite over a subring

$B = k[y_1, \dots, y_{n-1}]$ see the next lemma!

By induction B is finite over a subring

$C = k[z_1, \dots, z_d]$ with the z_i alg indep.

And A finite over C. $\square$

Lemma. Let $A = k[x_1, \dots, x_n]$ a fin. gen. k-alg.
Say $x_1, \dots, x_{n-1}$ alg indep, x_n not.
Then $\exists\, c_1, \dots, c_{n-1}$ s.t. A is finite over
$k[x_1 - c_1 x_n, \dots, x_{n-1} - c_{n-1} x_n]$.

Pf. Assumptions $\Longrightarrow \exists$ nonzero $f(X_1, \dots, X_{n-1}, T)$
s.t. $f(x_1, \dots, x_n) = 0$.
$x_1, \dots, x_{n-1}$ alg indep $\Longrightarrow T$ appears in f.
$\rightsquigarrow$ think of f as a poly in T:
$f(X_1, \dots, X_{n-1}, T) = a_m T^m + \dots + a_0$
$a_i \in k[X_1, \dots, X_{n-1}]$
example. $f = X_1 T^2 + T + X_2$
Do a change of variables $X_1 \longrightarrow X_1 + T$
$\rightsquigarrow g = (X_1 + T) T^2 + T + X_2$
$= T^3 + X_1 T^2 + T + X_2$
Now, $g(x_1 - x_n, x_2, \dots, x_n) = 0$
$\Longrightarrow x_n$ integral over $k[x_1 - x_n, x_2, \dots, x_{n-1}]$
$\Longrightarrow A$ finite over $k[x_1 - x_n, x_2, \dots, x_{n-1}]$.
$\square$

See Milne Lemma 2.43