Chapter 2: Morphisms

Polynomial Maps

Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ aff. alg. vars.

**Def.** $f: X \rightarrow Y$ is a morphism if it is the restriction of a polynomial map. That is, $\exists f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ s.t. $f(x) = (f_1(x), \ldots, f_m(x)) \forall x \in X$.

**Fact.** Morphisms are continuous in the Zariski topology.

**Pf.** $f^{-1}(Z(h_1, \ldots, h_r)) = Z(h_1 \circ f, \ldots, h_r \circ f)$.

**Def.** A morphism is an isomorphism if it has an inverse.

**Example.** An affine change of coordinates on $\mathbb{A}^n$ is a morphism:

$\mathbb{A}^n \rightarrow \mathbb{A}^n$

$x \mapsto (L_1(x), \ldots, L_n(x))$

$L_i(x) = \lambda_{i1}x_1 + \cdots + \lambda_{in}x_n + m_i$

This is invertible iff $(\lambda_{ij})$ is.
Example. $\mathbb{A}^2 \rightarrow \mathbb{A}'$

$(x,y) \mapsto x$

is a morphism. It is not an isomorphism since it is not invertible.

Example. $X = \mathbb{Z}(y-x^2) \subseteq \mathbb{A}^2$

$\mathbb{A}' \rightarrow X$

$t \mapsto (t,t^2)$

is an isomorphism, with inverse

$(t,t^2) \mapsto t$

Example. $X = \mathbb{Z}(y^2-x^3) \subseteq \mathbb{A}^2$

$f : \mathbb{A}' \rightarrow X$

$t \mapsto (t^2,t^3)$

is bijective, but not an isomorphism.

One candidate inverse is $(x,y) \mapsto y/x$.

How do we know this is not a polynomial?

We need a new tool for this!

The projection $\mathbb{A}^2 \rightarrow \mathbb{A}'$ shows that morphisms do not always map varieties to varieties. The hyperbola $X = \mathbb{Z}(xy-1) = \{(t,t^{-1}) : t \neq 0\}$ is a closed set and the image is $\mathbb{A}' \setminus \{0\}$, which is not closed.
The Coordinate Ring

Let \( X \subseteq \mathbb{A}^n \) be an aff. alg. var., \( f \in k[x_1, \ldots, x_n] \)

\[ \sim \text{ restriction } f|_X \]

The coordinate ring of \( X \) is

\[ k[X] = \{ f|_X : f \in k[x_1, \ldots, x_n] \} \]

\[ = \{ \text{polynomial fns on } X \} \]

\( k[X] \) is a ring, in fact a \( k \)-algebra. In fact:

\[ k[X] \cong k[x_1, \ldots, x_n]/I(X). \]

Example. Let \( X = \mathbb{Z}(xy-1) \). Then \( \frac{1}{x} \) lies in \( k[X] \) since it is equivalent to the polynomial \( y \).

Note. \( k[\mathbb{A}^n] = k[x_1, \ldots, x_n] \)

\[ k[p] \cong k \] (see the above proof that \( (x_1-a_1, \ldots, x_n-a_n) \) is maximal)

\[ k[p_1, \ldots, p_r] \cong k^r \]
**Example.** \( X = \mathbb{Z}(x^2+y^2-z^2) \leq \mathbb{A}^3 \) cone \( k = \mathbb{C} \)

\( x^2+y^2-z^2 \) irreducible (if not, it would be a product of 2 linear, homogeneous factors...)

\[ \Rightarrow (x^2+y^2-z^2) \text{ prime, hence radical} \]

\[ \Rightarrow \mathcal{I}(X) = (x^2+y^2-z^2) \quad (SN) \]

\[ \Rightarrow \mathbb{C}[X] = \mathbb{C}[x,y,z]/(x^2+y^2-z^2) \]

Can say: \( \mathbb{C}[x,y,z] \) equipped with the relation

\( x^2+y^2-z^2 = 0. \)

So:

\[ x^3+2xy^2-2xz^2+x \]

\[ = 2x(x^2+y^2-z^2)+x-x^3 \]

\[ = x-x^3 \]

**Example.** \( X = \mathbb{Z}(y-x^2) \)

Every \( f \in k[X] \) can be written as a poly.
in \( x \) (just replace all \( y \)'s with \( x^2 \)).

**Prop.** \( X \text{ irreducible} \iff k[X] \text{ an integral domain.} \)

\( \text{no } 0\text{-divisors} \)

**Pf.** \( R/J \text{ an int. dom.} \iff J \text{ prime.} \)

**Fact.** \( k[X] \) is the (fin. gen.) \( k \)-alg generated by the coordinate functions \( X \rightarrow k, x \mapsto x_i \).
Example. \( X = \mathbb{Z}(y-x^2, z-x^3) \) "twisted cubic"

Claim: \( I(X) = (y-x^2, z-x^3) \)

**Pf:** Let \( f \in I(X) \). Use div alg wrt \( y \), then \( z \rightleftharpoons f(x,y,z) = (y-x^2)g(x,y,z) + (z-x^3)h(x,y,z) + r(x) \)
Then \( \forall t \in k \), \( (t,t^2,t^3) \in X \), so \( r(t) = 0 \ \forall t \rightleftharpoons r = 0 \), whence the claim.

\[ \text{In } k[x] : \ y = x^2, \ z = x^3 \rightleftharpoons k[x] \cong k[x] , \]
\[ \text{an integral domain. } \Rightarrow X \text{ irred.} \]

Another proof: \( \mathbb{A}^1 \rightarrow X \ t \mapsto (t,t^2,t^3) \)
is a surjective morphism \( \Rightarrow X \text{ irred.} \)

**Dictionary**

As with all of \( \mathbb{A}^n \), there is a dictionary:

subvarieties \( \leftrightarrow \) radical ideals
\( Y \subseteq X \quad J \subseteq k[x] \)
irred subvarieties \( \leftrightarrow \) prime ideals
pts \( \leftrightarrow \) max ideals

**Pf. Homework!** 3rd isom. thm!
Pullback

A morphism \( f: X \rightarrow Y \) induces a pullback

\[ f_*: k[Y] \rightarrow k[X] \]

\[ g + I(Y) \mapsto g \circ f + I(X) \]

This is well def. since the composition of polynomials is a polynomial, and because if \( g \in I(Y) \) then \( g \circ f \) lies in \( I(X) \).

Note:
- \( f_* \) is a \( k \)-algebra homom.
- \( (fg)_* = g_*f_* \)
- \( f \) an isomorphism \( \Rightarrow f_* \) is

Example.

\[ \mathbb{A}^1 \rightarrow \mathbb{A}^2 \]

\[ t \mapsto (t, t^2) \quad \text{(already said this was \( \cong \))} \]

Pullback:

\[ \mathbb{C}[x,y]/(y-x^2) \rightarrow \mathbb{C}[t] \]

\[ x \mapsto t \]

\[ y \mapsto t^2 \]

surjective with trivial kernel, hence \( \cong \).
Example. \( \mathbb{A}^1 \rightarrow \mathbb{Z}(y^2 - x^3) \leq \mathbb{A}^2 \)
\[ t \mapsto (t^2, t^3) \]
Pullback:
\[ \mathbb{C}[x,y]/(y^2 - x^3) \rightarrow \mathbb{C}[t] \]
\[ x \mapsto t^2 \]
\[ y \mapsto t^3 \]
Not an \( \cong \) since \( t \) not in the image.
So the map \( \mathbb{A}^1 \rightarrow X \) is not \( \cong \).

Example. Is \( X = \mathbb{Z}(xy - 1) \) isomorphic to \( \mathbb{A}^1 \)?
No.
Have: \( k(\mathbb{A}^1) = k[x] \)
\[ k(X) = k[x, x^{-1}] \]
Laurent polynomials
Want to show these are not isomorphic.
Suppose \( \Phi: k[x, x^{-1}] \rightarrow k[x] \) is an isomorphism.
\[ \Rightarrow \Phi(1) = 1 \]
\[ \Rightarrow \Phi(x) \Phi(x^{-1}) = 1 \]
\[ \Rightarrow \Phi(x), \Phi(x^{-1}) \text{ units in } k[x] \]
\[ \Rightarrow \text{they are scalars} \]
\[ \Rightarrow \text{Im } \Phi \leq \text{scalars. CONTRADICTION.} \]
More Geometry vs. Algebra

An algebra is reduced if it has no nilpotent elts: \( r^n = 0 \).

\( \mathbb{K} = \text{alg. closed} \).

Thm (i) Every \( \mathbb{K}[X] \) is a fin. gen. red. \( \mathbb{K} \)-alg.

(ii) Every finitely gen. reduced \( \mathbb{K} \)-algebra is isom. to some \( \mathbb{K}[X] \).

(iii) If \( f: X \rightarrow Y \) is a morphism, then
\[ f_*: \mathbb{K}[Y] \rightarrow \mathbb{K}[X] \]

is a homomorphism.

(iv) If \( \sigma: R \rightarrow S \) is a homom. of reduced fin. gen. \( \mathbb{K} \)-algebras, then there is a morphism
\[ f: X \rightarrow Y \] with \( f_* = \sigma \). This \( f \) is unique up to isomorphism.

In other words the categories of aff. alg. var.'s & fin. gen. red. \( \mathbb{K} \)-alg's are (anti-)isomorphic.

Note. In 1950's Grothendieck removed 3 hypotheses: fin gen, red, alg closed. The corresponding geometric objects are affine schemes.
Pf. (i) \( k[X] \) is gen. by. images of the \( x_i \).
\( I(X) \) radical \( \Rightarrow \) reduced.

(ii) Let \( R \) be a fin. gen. red. \( k \)-alg.
Choose a “presentation.” If the gens are \( y_1, \ldots, y_m \),
then \( R = k[y_1, \ldots, y_m]/J \)
\( (J \text{ is kernel of } k[y_1, \ldots, y_m] \to R) \)
\( R \text{ reduced } \Rightarrow J \text{ radical} \)
Let \( Y = Z(J) \). \( S_{N} \Rightarrow k[Y] = R \).

(iii) We already know this.

(iv) Fix \( \sigma : R \to S \). As above:
\( \sigma : k[y_1, \ldots, y_m]/J \to k[x_1, \ldots, x_n]/I \)
Again, \( I \) & \( J \) radical.
\( \Rightarrow R, S \) are coord rings of \( Z(J) \) & \( Z(I) \).
Want polynomial \( f : \mathbb{A}^n \to \mathbb{A}^m \)
s.t. \( Z(I) \hookrightarrow Z(J) \)
& \( f_* = \sigma \)
Let $\tilde{\tau} : k[y_1, \ldots, y_m] \rightarrow S$
\[ f \mapsto \tilde{\tau}(f) \]

This is a $k$-alg. homom. (it's the composition of 2 such)

Let $f_i = \text{rep. of } \tilde{\tau}(y_i)$
i.e. $\tilde{\tau}(y_i) = f_i + J$

Define $f : A^n \rightarrow A^m$
\[ x \mapsto (f_1(x), \ldots, f_n(x)) \]

**Claim 1.** $f(Z(I)) \subseteq Z(J)$

**Pf.** $\forall x \in Z(I) \text{ want } f(x) \in Z(J)$
i.e. $g \circ f(x) = 0 \ \forall g \in J, x \in Z(I)$
i.e. $g \circ f \in I$

Fix a $g \in J$. Have:

$g \circ f + I = g(f_1, \ldots, f_n) + I$

These terms make sense.
They are independent of choice of rep of $\tilde{\tau}(x_i)$.

\[
\begin{aligned}
&= g(\tilde{\tau}(x_1), \ldots, \tilde{\tau}(x_n)) + I \\
&= \tilde{\tau}(g(x_1, \ldots, x_n)) + I \\
&= \tilde{\tau}(g) + I \\
&= 0 + I \quad \checkmark
\end{aligned}
\]
Claim 2. \( f_* = \emptyset \)

**Pf.** Let \( g \in k[y_1, \ldots, y_m] \)
\[
 f_*(g + J) = g \circ f + I = \mathcal{F}(g) + I \quad \text{(as above)}
 = \sigma(g + J)
\]

We really should have done this claim first.
The previous claim is the special case \( f_* (0) = \emptyset (0) \)

Claim 3. \( f \) is unique: If \( f, g : X \to Y \) have
\[
 f_* = g_* \quad \text{then} \quad f = g.
\]

**Pf.** Write \( I(Y) = k[y_1, \ldots, y_m] / J \).

Have \( f^*(y_i + J) = g^*(y_i + J) \)
\[\sim y_i \circ f + I = y_i \circ g + I \]
Say \( f = (f_1, \ldots, f_m) \quad g = (g_1, \ldots, g_m) \)
\[\sim f_i + I = g_i + I \]
\[\sim f + I = g + I \]
i.e. \( f|_X = g|_X \quad \checkmark \)
There is a loose end: we didn't show that the \(X\) & \(Y\) we constructed are unique.

**Prop.** \(X \subseteq A^n, Y \subseteq A^m\) aff. alg. var's. 
\(f: X \rightarrow Y\) a morphism. 
Then \(f\) is an isomorphism \(\iff f^*\) is.

**Pf.** 
\(\Rightarrow\) done already.
\(\Leftarrow\) \(f^*\) an iso
\(\Rightarrow\) \(\exists \sigma: k[X] \rightarrow k[Y]\)
s.t. \(f^* \circ \sigma = \text{id} \& \sigma \circ f^* = \text{id}\).
Any such \(\sigma\) is \(g^*\) for some \(g: Y \rightarrow X\)
(We gave the argument above. It's just that instead of starting with random \(R\) & \(S\) we start with \(k[X], k[Y]\)).
Now: \(f^* g^* = (gf)^* = \text{id}\)
\(\Rightarrow\) \(gf = \text{id}\) (Claim 3).

**Cor.** \(X, Y\) aff. alg. vars.
Then \(X \cong Y \iff k[X] \cong k[Y]\)

**Pf.** 
\(\Rightarrow\) done.
\(\Leftarrow\) Any \(k[X] \rightarrow k[Y]\) gives \(Y \rightarrow X\) as above
Apply the Prop. □
**Dictionary (= Functor)**

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>aff. alg. var.</td>
<td>fin gen. red. k-alg R</td>
</tr>
<tr>
<td>alg. subset</td>
<td>rad. ideal in R</td>
</tr>
<tr>
<td>irred. alg. subset</td>
<td>prime ideal in R</td>
</tr>
<tr>
<td>point</td>
<td>max ideal in R</td>
</tr>
<tr>
<td>poly. map</td>
<td>k-alg homom.</td>
</tr>
</tbody>
</table>

For a more organized exposition of this last theorem, see Moraru.
**Dimension**

\[ X = \text{aff. alg. var.} \]

**Def** \( \dim X = \text{supremum of lengths of chains} \)
\[ X > X_1 > \ldots > X_d \] of distinct
 \text{irred. aff. alg. var.'s.} \]

**Fact.** \( \dim X = \max \dim X_i \) where \( \{X_i\} \) are the
irred. components.

**Fact.** If \( X \subseteq Y \) then \( \dim X \leq \dim Y \).

\[ \text{So: } \dim X = 0 \iff X = \text{pt.} \]

By the above dictionary:
\[ \dim X = \text{krull dim of } k[X]. \]

Some names:

- 0-dim pts
- 1-dim curve
- 2-dim surface
- n-dim n-fold
Problem. What is dim \( \mathcal{M}^n \)?

Obviously, \( \dim \mathcal{M}^n \geq n \). Will (almost) prove:

\[ \text{Thm.} \quad \dim X = \text{transc. deg}_k k(X) \]

\[ \text{Def.} \quad \text{For a comm. ring } A, \ x \in A \]

\[ S_{\{x\}} = \{ x^n (1 - ax) : n \in \mathbb{N}, a \in A \} \]

\[ \text{[This is a multiplicative set. Check this!]} \]

The boundary \( A_{\{x\}} \) of \( A \) at \( x \) is the ring of fractions \( S_{\{x\}}^{-1} A \).

\[ \text{Fact 1.} \quad S = \text{mult. subset of a ring } A \]

\[ \{ \text{prime ideals disjoint from } S \} \leftrightarrow \{ \text{prime ideals in } S^{-1} A \} \]

\[ \rho \rightarrow S^{-1} \rho = (S^{-1} A) \rho \]

inverse image \( \leftarrow q \)

\[ \text{Pf.} \quad \text{Milne 1.14} \]
Fact 2. \( A = \text{ring, } \forall x \in A, \text{ max ideal } m \triangleleft A \quad m \cap S_{\{x\}} \neq \emptyset. \)

**Pf.** If \( x \in m \) then \( x = x' (1-ox) \in S_{\{x\}} \)

If \( x \not\in m \) then \( x \) is invertible mod \( m \)

\[ \Rightarrow \exists a \text{ s.t. } 1-ax \in m \tag{\Box} \]

Fact 3. \( A = \text{ring, } m \triangleleft A \text{ max. ideal, } p \triangleleft m \text{ prime.} \)

\[ \forall x \in m \setminus p, \ p \cap S_{\{x\}} = \emptyset. \]

**Pf.** Suppose not: \( x^n(1-ax) \in p. \)

\[ \Rightarrow 1-ax \in p \Rightarrow 1-ax \in m \Rightarrow 1 \in m \tag{\Box} \]

Recall: Krull dim = max length of a chain of prime ideals.

Prop. \( A = \text{ring, } n \in \mathbb{N}. \)

\[ \text{Krull dim } A \leq n \iff \forall x \in A, \text{ Krull dim } A_{\{x\}} \leq n-1 \]

**Pf.** Fact 1: \( \{\text{prime ideals} \} \quad \text{disjoint from } S_{\{x\}} \}

\[ \iff \{\text{prime ideals} \} \quad \text{in } A_{\{x\}} \}

Fact 2: A chain of prime ideals beginning with a max ideal gets shortened in \( A_{\{x\}}. \)

Fact 3: Such a chain gets shortened by at most 1. \( \Box \)
Prop. \( A \) = integral domain.
\( F(A) \) = field of fractions
\( k \subseteq A \) subfield.
Then \( \text{tr. deg}_k F(A) \geq \text{Krull dim } A \)

Pf. WLOG \( k \) alg. closed.
If \( \text{tr. deg} = \infty \), nothing to prove.
Say \( \text{tr. deg}_k F(A) = n \). Induction.
Let \( x \in A \).
\( \cdot \) If \( x \notin k \), then \( x \) transc. over \( k \)
\[ \Rightarrow \text{tr. deg}_{k(x)} F(A) = n - 1 \]
Since \( F(A) \cong F(A[x]) \), have: \( \text{tr. deg}_{k(x)} F(A[x]) = n - 1 \)
Since \( k(x) \subseteq A[x] \), induction gives
\( \text{Krull dim } A[x] \leq n - 1 \).
Previous Prop \( \implies \text{dim } A \leq n \).
\( \cdot \) If \( x \in k \), then \( 0 = 1 - x'x \in S[x] \Rightarrow A[x] = 0 \).
Again \( \text{dim } A[x] \leq n - 1 \) \( \square \)

Cor. \( \text{Krull dim } k[x_1, \ldots, x_n] = n \).
Pf. \( \geq \) \( (x_1, \ldots, x_n) \supseteq (x_1, \ldots, x_{n-1}) \supseteq \cdots \supseteq (x_1) \supseteq 0 \)
\( \leq \) Previous Prop. \( \square \)

Cor. \( \text{dim } A^n = n \).
HYPERSURFACES

Prop. A hypersurface in $\mathbb{A}^n$ has dim $n-1$.

Pf. Let $H$ be a hypersurface.

WLOG $H$ irreducible.

$\Rightarrow H = \mathbb{H}(f)$, $f$ irreducible.

Let $k[x_1,\ldots,x_n] = k[x_1,\ldots,x_n]/(f)$, $x_i = x_i + (f)$ and $k(x_1,\ldots,x_n)$ the field of fractions.

$f \neq 0 \Rightarrow$ some $x_i$, say $x_n$, appears in it.

$\Rightarrow$ no nonzero poly in $x_1,\ldots,x_{n-1}$ lies in $(f)$.

$\Rightarrow x_1,\ldots,x_{n-1}$ alg indep.

But $x_n$ is alg. over $x_1,\ldots,x_{n-1}$ (think of $f$ as a poly. in $x_n$ w/ coeffs in $k[x_1,\ldots,x_{n-1}] \subseteq k(x_1,\ldots,x_n)$).

$\Rightarrow \{x_1,\ldots,x_{n-1}\}$ is a trans. basis for $k(x_1,\ldots,x_n)$ over $k$. Apply the theorem. \qed

Example. Say $f(x,y), g(x,y)$ nonconstant, no common factors.

Then $\dim \mathbb{Z}(f) = 1$ by (2).

Also: $\dim \mathbb{Z}(f, g) < \dim \mathbb{Z}(f)$

$\Rightarrow \mathbb{Z}(f, g)$ = finite set of points.

How many? Stay tuned (Bézout).
Prop. The closed sets of codim 1 in $\mathbb{A}^n$ are exactly the hypersurfaces.

Pf. Say $W = \text{aff. alg. var of codim 1}$. $W_1, \ldots, W_s$ the irreducible components. $I(W) = \cap I(W_i)$, so if $I(W_i) = Z(f_i)$ then $I(W) = Z(f_1 \cdots f_r)$. Thus, WLOG $W$ irreducible.

$I(W)$ is prime, nonzero. Let $f$ be an irreducible poly in $I(W)$.

$\Rightarrow$ $(f)$ prime.

If $(f) \neq I(W)$ then

$I(W) \supseteq f \supseteq (0)$ distinct primes.

$\Rightarrow \mathbb{A}^n \supseteq Z(f) \supseteq W$

$\Rightarrow \text{codim } W > 1$. $\square$

Classification of Irred. Aff. Alg. Vars in $\mathbb{A}^2$

$\text{dim } 2 : \mathbb{A}^n \leftrightarrow (0)$

$\text{dim } 1 : \text{hypersurfaces} \leftrightarrow (f) \ f \text{ irreducible}$

$V = V(f)$, where $f$ is any irreducible in $I(V)$.

$\text{dim } 0 : \text{pt.} \leftrightarrow (x_1-a_1, x_2-a_2)$
**Noether Normalization**

Say a $k$-alg. $B$ is finite over a $k$-alg $A$ if there are $b_1, \ldots, b_n$ s.t. $A$-span of the $b_i$ is $B$.

e.g. $k[x]$ is finitely gen. over $k$ but not finite over $k$.

Say $b \in B$ is integral over $A$ if
$$b^n + a_{n-1}b^{n-1} + \ldots + a_0 = 0.$$ 

**Fact.** $b$ integral over $A \iff A[b]$ finite over $A$.

**Thm.** $A$ fin gen. $k$-alg. $\exists x_1, \ldots, x_d \in A$ alg indep. over $k$ s.t. $A$ is finite over $k[x_1, \ldots, x_d]$.

**Pf.** (assuming $k$ infinite)

Let $A = k[x_1, \ldots, x_n]$ (really, a quotient of this).

Induct on $n$.

If $\{x_i\}$ alg indep, nothing to prove.

Otherwise, you show $A$ is finite over a subring $B = k[y_1, \ldots, y_{n-1}]$ see the next lemma!

By induction, $B$ is finite over a subring $C = k[z_1, \ldots, z_d]$ with the $Z_i$ alg indep.

And $A$ finite over $C$. $\Box$
Lemma. Let $A = k[x_1, \ldots, x_n]$ a fin. gen. $k$-alg. Say $x_1, \ldots, x_{n-1}$ alg indep, $x_n$ not. Then $\exists c_1, \ldots, c_{n-1}$ s.t. $A$ is finite over $k[x_1 - c_1 x_n, \ldots, x_{n-1} - c_{n-1} x_n]$.

Pf. Assumptions $\Rightarrow \exists$ nonzero $f(x_1, \ldots, x_{n-1}, T)$ s.t. $f(x_1, \ldots, x_n) = 0$. $x_1, \ldots, x_{n-1}$ alg indep $\Rightarrow T$ appears in $f$. $\Rightarrow$ think of $f$ as a poly in $T$: $f(x_1, \ldots, x_{n-1}, T) = a_m T^m + \cdots + a_0$ $a_i \in k[x_1, \ldots, x_{n-1}]$ example. $f = x_1 T^2 + T + x_2$ Do a change of variables $x_1 \mapsto x_1 + T$ $\Rightarrow g = (x_1 + T)T^2 + T + x_2$ $= T^3 + x_1 T^2 + T + x_2$ Now, $g(x_1 - x_n, x_2, \ldots, x_n) = 0$ $\Rightarrow x_n$ integral over $k[x_1 - x_n, x_2, \ldots, x_{n-1}]$ $\Rightarrow A$ finite over $k[x_1 - x_n, x_2, \ldots, x_{n-1}]$. □

See Milne Lemma 2.43