# Algebraic Geometry

Math 6421

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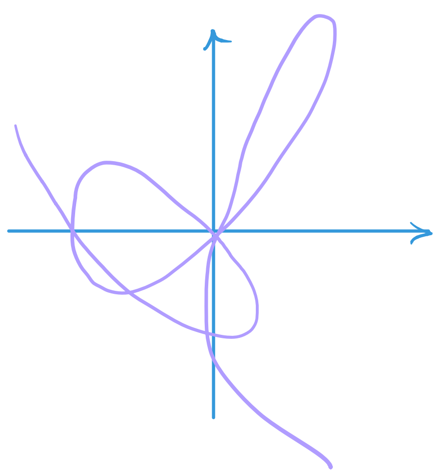
# What is Algebraic Geometry

In short: the study of solutions to polynomial equations.

So in a sense, it is the next thing after linear algebra.

Bad news: unlike linear algebra, even the set of solns to a single equation can be complicated:



Good news: taking derivatives is easy, and you can use the usual formulas over any field, even a finite one. Often the analogy works, even though it has no right to (cf. Weil conjectures).

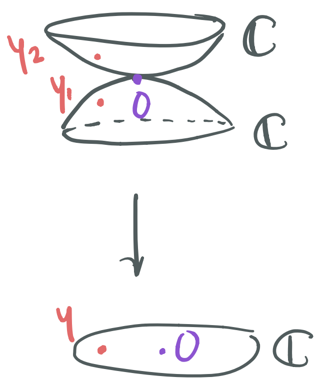
$\rightsquigarrow$ can define tangent spaces, cohomology... for <u>non</u>-manifolds!

# First Examples

Consider: $y^2 = x$ $\qquad x, y \in \mathbb{C}$

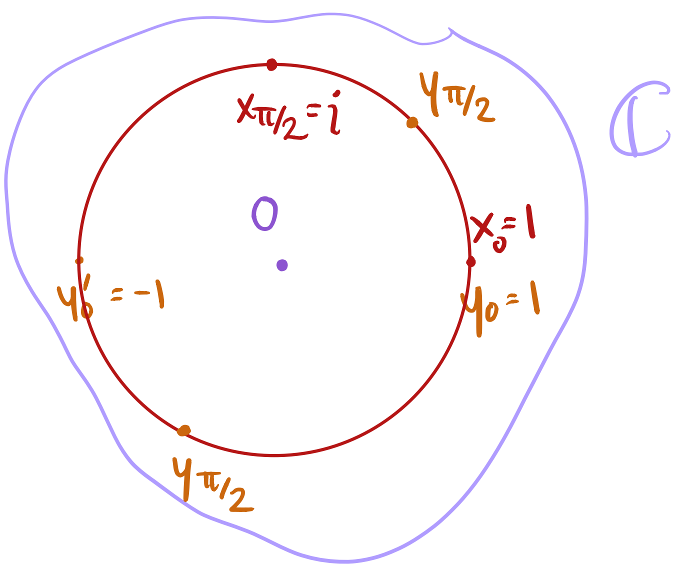
$\rightsquigarrow$ 2 copies of $\mathbb{C}$ identified at $0$.

But it's not:



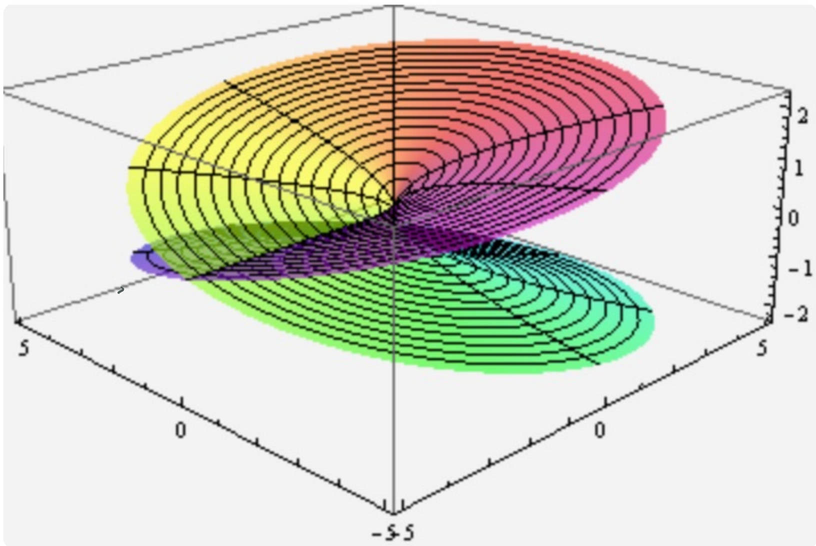

Why not? The above picture suggest there exists a "first" and "second" root of $x$.

But that's not what really happens:

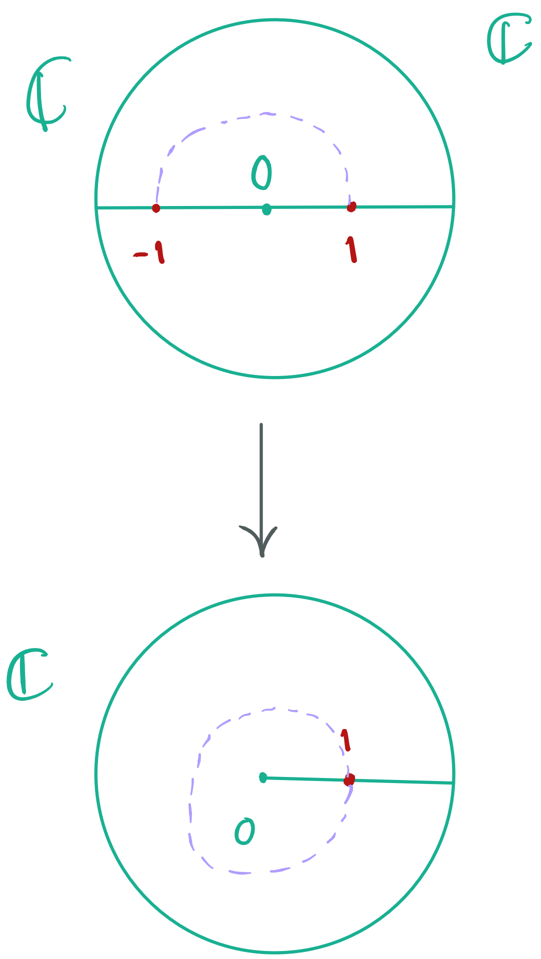



If you keep track, the roots of $x$ swap places as $x$ travels around $S^1$.

A better picture:




Or:



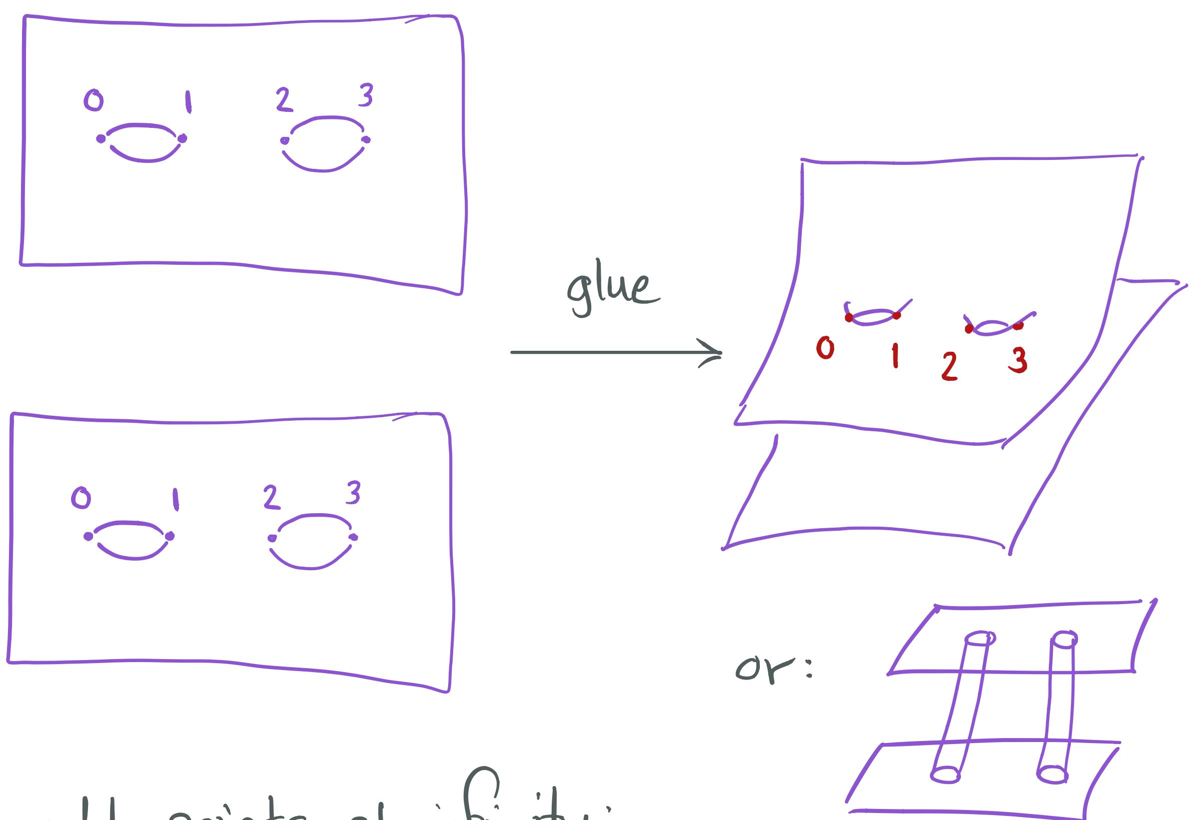

Can think of making the top picture by cutting the bottom one along any arc from 0, taking 2 copies of the cut-open disk, and gluing.

The slit is a clever way of recording the number of times you wind around zero.

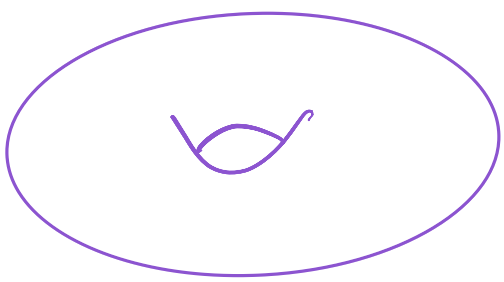
A more interesting example:

$$y^2 = x(x-1)(x-2)(x-3)$$

Should again have 2 copies of $\mathbb{C}$, identified at 0, 1, 2, 3. Have a similar phenomenon: a small loop around any of these 4 pts gets you from one copy to the other. Here's how you do it:




Can add points at infinity:

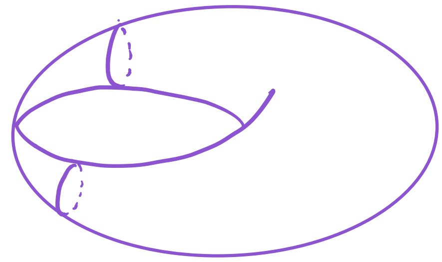


Can make the example harder:

$$y^2 = x(x-t)(x-2)(x-3)$$

What happens as $t$ varies? Say it approaches 0.

One of the tubes shrinks to a point:



"singular pt"

"node"

Still has genus 1 (whatever that means).

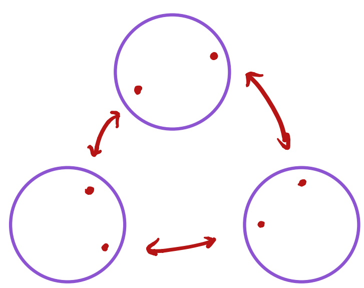
More generally, if we have $f(x,y)=0$ where $f$ has degree $d$, we can degenerate (as above) to the case where $f$ is a product of $d$ linear eqns:

$$\ell_1(x,y) \cdots \ell_d(x,y) = 0.$$

(how?)

The set of solutions is a set of $d$ lines. Any two intersect in one pt. Can arrange so no 3 intersect:

$d=3$



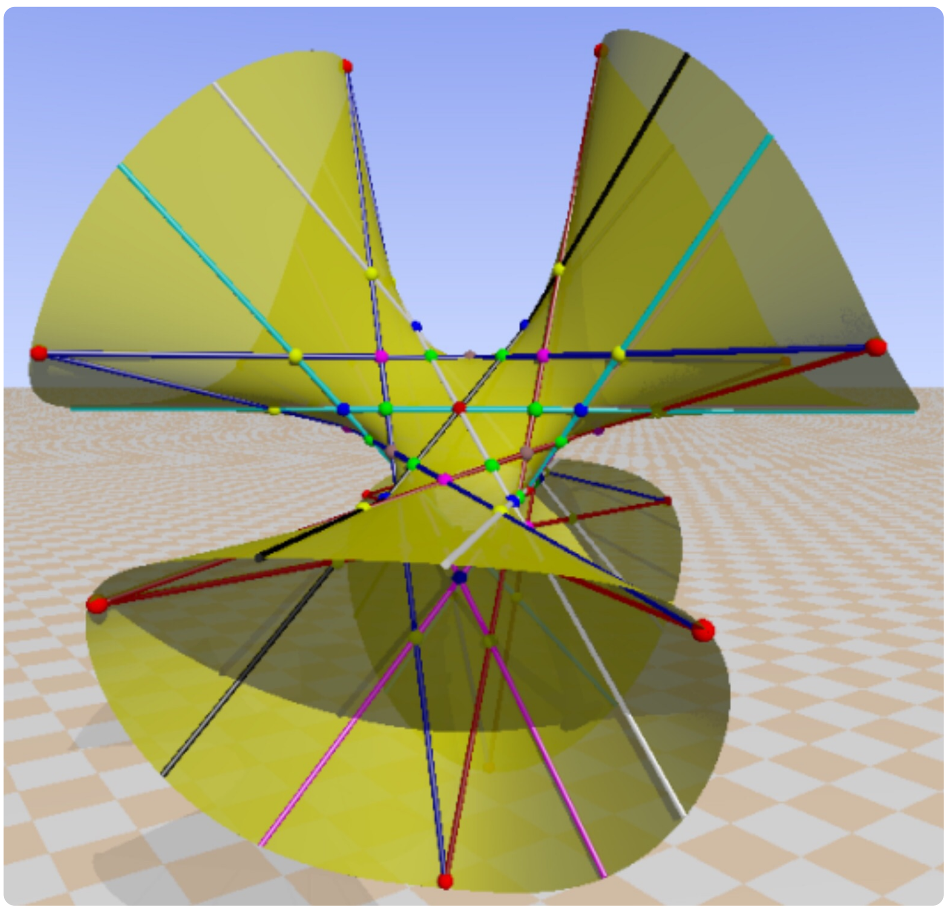
$\rightsquigarrow g=1.$

This agrees with the degree-genus formula for plane curves:

$$g = \binom{d-1}{2}$$

Moral: degenerations (to non-manifolds) are advantageous.

A famous theorem. Every cubic surface (so, 3 vars & 1 eqn) contains exactly 27 lines. "enumerative geometry"



**Another famous theorem.** $x^n + y^n = 1$ has no solutions in $\mathbb{Q}$ when $n \geq 3$.

Over $\mathbb{C}$ there are many solutions. By the degree-genus formula, the solution set has genus $\binom{n-1}{2}$. The proof uses the theory of algebraic curves.