Schemes are the main objects of study in algebraic geometry. The main developments are due to Grothendieck in the 1960's.

The (very) basic idea is this: instead of starting with a space $X$ and obtaining a ring $\mathcal{O}_X(X)$, we start with an arbitrary ring $R$ and create a space $\text{Spec}(R)$.

The ring $R$ might have nilpotent elements. We can use these to record higher order intersections. Consider the intersection $Z(y-x^2) \cap Z(y)$. Normally we eliminate $y$ to obtain $Z(x^2) \subseteq \mathbb{A}^1$ then take radical to get $Z(x)$. With schemes, we leave it as $Z(x^2)$, yielding a nilpotent element that records the second order intersection.

One consequence is that there is a Bézout theorem that holds all the time, not just generically.
Another thing that happens in scheme theory is that we can treat varieties over finite fields using geometric intuition from $\mathbb{C}$. We'll see $\text{Spec}(\mathbb{Z})$ consists of $0 \cup \{\text{primes}\}$. Given an algebraic curve with $\mathbb{Z}$ coefficients, we can reduce mod $p$, yielding a family of "curves", one for each $p$. Scheme theory allows us to relate these to each other. (Cf. Weil conjectures).

**Example: the Frobenius map**

We write any polynomial, say with $\mathbb{Z}$-coeffs:

$$f(x) = x^2 - x + 3$$

What are the roots in $k$, for various $k$?

How many roots does it have in $F_7$?

Let $k = \overline{F_p}$

$F_p : \mathbb{A}^n \to \mathbb{A}^n$

$$x_i \mapsto x_i^p$$

This is a bijection (why?) but not an isomorphism, since $F_p^* : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ is not surjective ($x_i$ is not in the image).
Fact. For \( m \geq 1 \), \( \mathbb{F}_p^m \) is the unique subfield of \( k \) with degree \( m \) over \( \mathbb{F}_p \). And it equals the set of fixed pts of \( \mathbb{F}_p^m \).

Say now \( f(x_1, \ldots, x_n) \) is a polynomial with coeffs in \( \mathbb{F}_p^m \), say
\[
f = \sum \mathcal{C}_I X^I = \mathcal{C}_{i_1, \ldots, i_n} X^{i_1} \cdots X^{i_n}, \quad \mathcal{C}_I \in \mathbb{F}_p^m.
\]

If \( (a_1, \ldots, a_n) \in \mathbb{Z}(f) \subseteq \mathbb{A}^n_k \) then
\[
0 = \mathbb{F}_p^m \left( \sum \mathcal{C}_I a_{i_1}^{i_1} \cdots a_{i_n}^{i_n} \right) = \sum \mathbb{F}_p^m(a_{i_1}^{i_1}) \cdots \mathbb{F}_p^m(a_{i_n}^{i_n})
\]
\[
\Rightarrow \mathbb{F}_p^m(a_1, \ldots, a_m) \subseteq \mathbb{Z}(f).
\]

So \( \mathbb{F}_p^m \) maps \( \mathbb{Z}(f) \) to itself. And the fixed points are the points in \( \mathbb{F}_p^m \).

We wanted to count these. In algebraic topology, we would use the Lefschetz fixed point theorem.

How can we do that here? \( \mathbb{Z}(f) \) is a discrete set! Answer: define schemes (rings with a topology on their set of prime ideals...), then define étale cohomology for schemes, then come up with a Lefschetz fixed point theorem, solve the Weil conjectures, win the Fields medal...
\( R = \text{commutative ring} \)

**Def.** The prime spectrum, or spectrum, of \( R \) is the collection of prime ideals, denoted \( \text{Spec}(R) \).

We also refer to \( \text{Spec}(R) \) as the affine scheme associated to \( R \).

Before, points of \( X \) corresponded to max. ideals in \( k[X] \). So \( \text{Spec}(R) \) has “extra” points. Some of these “points” are contained in others.

**Note.** 
- \( R \) itself is not prime.
- \( 0 \) is prime iff \( R \) is a domain.

We should think of: \( \text{Spec}(R) \leftrightarrow X \)

\[ R \leftrightarrow k[X] \]

Because of this, we’ll want to think of elements of \( R \) as functions on \( \text{Spec}(R) \).
To this end: For each $p \in \text{Spec}(R)$ let $k(p)$ denote the quotient field of $R/p$.

Here is how $f \in R$ can be thought of as a function on $\text{Spec}(R)$: for $p \in \text{Spec}(R)$, let $f(p)$ be the image of $f$ under $R \to R/p \to k(p)$.

We call $f(p)$ the value of $f$ at $p$.

Note that these values lie in different fields: the function $5$ on $\text{Spec } \mathbb{Z}$ takes the value $1 \text{ mod } 2$ at $(2) \in \text{Spec } \mathbb{Z}$ and $2 \text{ mod } 3$ at $(3)$.

The statement $f \circ p$ translates to $f(p) = 0$.

The fact that we can add & multiply functions pointwise translates to the fact that $R \to k(p)$ is a ring homomorphism.

Will eventually interpret these functions as global sections of the structure sheaf on $\text{Spec}(R)$. 
If $R = k[X] = k[x_1, \ldots, x_n]/\mathbb{I}(x)$ & $p$ is a max. ideal of $R$ (ie a pt of $X$), then $k(p) = k$ and the value of $f \in R$ is the value in the classical sense.

**Example.** Spec$(C[x]) = 0 \cup \{x - a : a \in C\}$

This is the full set of prime ideals since every polynomial factors into linear terms. Let’s call this space $A_C^n$

Picture:

![Diagram](http://example.com/diagram.png)

The functions on $A_C^n$ are polynomials. So $f(x) = x^2 - 3x + 1$ is a function. Its value at $(x-1)$ (which we think of as 1) is $f(1)$. Really we should take the equiv. class of $f(x)$ in $(C[x]/(x-1))$, but this is the same as setting $x = 1$. (by the division alg)

The value of $F$ at $(0)$ is just $f(x)$. Let try it!

This whole discussion works over any alg. closed $k$. 
Example. $\text{Spec } \mathbb{Z} = 0 \cup \{\text{primes}\}$

Same picture:

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \]

(2) (3) (5) (0)

It boils down to the fact that the intersection of all prime ideals is not 0.

100 is a function. Its value at 3 is 1. It has a (double!) Zero at 2...

Example. $\text{Spec } k = pt.$

Example. $R = k[\varepsilon]/\varepsilon^2$ “ring of dual numbers” $k$ alg. closed

Think of $\varepsilon$ as a small number (its square is 0).

Will show: $\text{Spec}(R) = \{ (\varepsilon) \}.$

Indeed: Primes of $R \leftrightarrow$ primes $\mathfrak{p} \subseteq k[x], \mathfrak{p} \ni (\varepsilon^2)$

$k[\varepsilon]$ principal so

$\mathfrak{p} = (f)$ and $(\varepsilon^2) \subseteq f \iff f | \varepsilon^2 \quad \square$

The function $\varepsilon$ is nonzero but its value at all points of $\text{Spec}(R)$ is 0. So:

functions are not determined by their values
Example. $\text{Spec}(R[x]) = \{(0)\} \cup \{(x-a)\} \cup \{\text{irred. quadratics}\}$
Call it $\mathbb{A}^1_R$
The first two pieces are familiar. The new pts are complex conjugate pairs.

Consider $f(x) = x^3 - 1$.
Its value at $(x-2)$ is 7, or $7 \mod (x-2)$; this is $f(2)$.
Its value at $(x^2 + 1)$ is $-x-1 \mod (x^2+1)$, which we can think of as $-i-1$.

Example. $\mathbb{A}^1_{\mathbb{F}_p} = \text{Spec} \mathbb{F}_p[x] = \{(0)\} \cup \{\text{irred. polys}\}$
(since $\mathbb{F}_p[x]$ is a domain.
Can identify each irred poly with the corresponding set of Galois conjugates in $\overline{\mathbb{F}_p}$.
A polynomial $f$ is not determined by its values on $\mathbb{F}_p$ but is det. by values on $\overline{\mathbb{F}_p}$

\text{e.g. } f(x) = 1 \& g(x) = x^2 + x + 1 \text{ with } p = 2.

Example. $\mathbb{A}^2_{\mathbb{C}} = \text{Spec} \mathbb{C}[x,y]$
Note: $\mathbb{C}[x,y]$ not principal: $(x,y)$ is not princ.
$(0) \in \mathbb{A}^2_{\mathbb{C}}$
$(x-a, y-b) \in \mathbb{A}^2_{\mathbb{C}}$ (these are max. ideals)
Other irreducibles also lie in $\mathbb{A}^2_C$, such as $y - x^2$ & $y^2 - x^3$.

To picture this, the $(x-a, y-b)$ correspond to points of $C^2$.

What about the "bonus" points? $(0)$ is "behind" all the traditional pts. It does not lie on $y = x^2$.

$(y - x^2)$ lies on $y = x^2$, but nowhere particular on it.

\[ \begin{array}{c}
\text{1-dim} \\
\downarrow \\
\text{f(x,y)} \\
\text{"closed pt"} \\
(x-a, y-b) \\
(0) \text{ 2-dim} \\
\end{array} \]

Example \[ \mathbb{A}^3_C = \text{Spec } C[x,y,z] \]

Again there are points of dim...

0: $(x-a, y-b, z-c)$
3: $(0)$
2: $(f)$ \( f \) irred.
1: impossible to classify; irred. curves example: $(x,y) \leftrightarrow z$-axis
**From Ring Operations to Spec Operations**

**Quotients.** \( \text{Spec } R/I \subseteq \text{Spec } R \)

Special case: \( S \subset R \) is a fin. gen. \( \mathbb{C} \)-alg. gen. by \( x_1, \ldots, x_n \) with relations \( f_i(x_1, \ldots, x_n) = 0 \). So \( R = k[x_1, \ldots, x_n]/(f_i) \) and \( \text{Spec } R/I \) is the set of pts of \( \text{Spec } R \) satisfying the \( f_i \), e.g.

**Localizations.** \( \text{Spec } S^{-1}R \subseteq \text{Spec } R \)

**Exercise:** \( \text{Spec } S^{-1}R \leftrightarrow \) primes in \( \text{Spec } R \) not meeting \( S \).

**Example.** \( S = \{1, f, f^2, \ldots \} \subseteq R \)

\[ \text{Spec } S^{-1}R = \{ p \in \text{Spec } R : f \notin p \} \]

**More specific.** \( R = \mathbb{C}[x, y] \) \( f(x, y) = y - x^2 \)

\[ \text{Spec } S^{-1}R = \{ x \in \text{Spec } R : f \text{ doesn't vanish} \} \]

= \( \mathbb{A}_\mathbb{C}^2 \) minus pts on \( y - x^2 \)

and the bonus pt \((y - x^2)\).
Vanishes everywhere iff it is nilpotent.

Geometrically: a function on Spec R.

The primes.

Thus, the nilradical is the intersection of all nilradicals. The set of nilpotents form an ideal called

\[(a, b, b^2) \mapsto (x, y^2, z^2)\]

\[\mathbb{C}[x, y, z]/(b - a^2) \xrightarrow{\text{Spec}} \text{Spec } \mathbb{C}[x, y, z]/(b - a^2, y^2 - x^2, h - x^2)\]

Say \( f : p \rightarrow C \)

\[C = \{(x, y, z) : z = y^2, h = x^2, x^2 = 0\} \subset \mathbb{C}\]

\[p = \{(a, b) : b = a^2\} \subset \mathbb{C}\]

Explicit example. f: B --> A map of rings
Topography

\( R = \text{comm. ring} \)
\( S \subseteq R \text{ subset} \)

\[ Z(S) = \{ p \in \text{Spec} R : f(p) = 0 \ \forall f \in S \} \]

As usual, the closed sets in \( \text{Spec} R \) are defined to be the \( Z(S) \)'s. This is the Zariski topology.

By the definition of value, we also have:

\[ Z(S) = \{ p \in \text{Spec}(R) : f \in p \ \forall f \in S \} = \{ p \in \text{Spec}(R) : p \supseteq S \} \]

As usual: \( Z(S) = Z((S)) \& S \subseteq T \Rightarrow Z(T) \subseteq Z(S) \)

**Example.** \( Z(xy, yz) \subseteq \mathbb{A}^3_{\mathbb{C}} = \text{Spec} \ \mathbb{C}[x, y, z] \)

This is the set of pts with \( y = 0 \) or with \( x = z = 0 \). Also, the bonus points:

- the generic point of the \( xz \)-plane, aka \((y)\)
- and the gen. pt of \( y \)-axis, aka \((x, z)\)

Also: 1-dim pts in \( xz \)-plane.
The $\mathcal{Z}(S)$ are the closeds for a topology on $\text{Spec}(R)$ since:

(i) $\bigcap \mathcal{Z}(I_i) = \mathcal{Z}(\sum I_i)$
(ii) $\mathcal{Z}(I) \cup \mathcal{Z}(J) = \mathcal{Z}(IJ)$
(iii) $\mathcal{Z}(I) \subseteq \mathcal{Z}(J) \iff I \subseteq J$

Example. \(\mathbb{A}_C^1\)

The open sets are: \(\emptyset\), \(\mathbb{A}_C^1\) minus a finite set of max ideals

Indeed, given $f \in C[X]$, we factor it $f = \prod (x - a_i)$

So $f \in p_i$ where $p_i = (x - a_i)$. Also, $f(0) \iff f = 0$ and $f$ contained in no prime ideals $\iff f$ const.

So: open sets are determined by their intersections with the traditional pts.

Example. $\text{Spec } \mathbb{Z}$

The open sets are $\emptyset$ & complement of finitely many ordinary primes.
Example, $\mathbb{A}^2_c$

Recall the pts are:

max ideals $(x-a,y-b)$ 0-dim

$(f(x,y))$ irred. 1-dim

$(0)$ 2-dim

The closed sets are:

- the whole space = closure of $(0)$
  \[ f \text{ vanishes on } (0) \implies f=0 \]
- a finite (possibly empty) set of curves
  (each the closure of a 1-dim pt)
  and finite number of 0-dim pts

To prove this, the hint is: if $f(x,y)$ and $g(x,y)$ are irred. poly's that are not multiples of each other their 0-sets intersect in a finite # of pts (this follows from fact that dim $\mathbb{A}^2_c = 2$, proved a long time ago).

\[ f: B \to A \]
\[ \sim f^* : \text{Spec } A \to \text{Spec } B \quad \text{continuous} \]

i.e. Spec is a contravariant functor Rings $\to$ Top.
Basis for the topology: For \( f \in R \),
\[
D(f) = \{ p \in \text{Spec}(R) : f(p) \neq 0 \}
\]

Fact. \( D(f) \subseteq D(g) \iff f^n \in (g) \) some \( n \neq 1 \)
\[ \iff g \text{ invertible in } A_f \]

Proof idea. \( Z(g) \leftrightarrow \text{Spec} (R/(g)) \)
\[ D(g) = Z(g)^c \]
\[ \implies f = \text{zero function on } Z(g) = \text{Spec } R/(g) \]
\[ \implies f \text{ nilpotent on } R/(g) \]
\[ \text{i.e. } f^n \in (g). \]

Def. In a top. space, we say a point is
- closed if it is its own closure
- generic if its closure is the whole space.
- generic in a closed set \( K \) if its closure is \( K \).

We say \( x \) is a specialization of \( y \) if \( x \in \overline{\{y\}} \)

eg \( (x-7, y-49) \) is a specialization of \( (y-x^2) \).

Fact. The closed pts of \( \text{Spec} R \) are the max ideals.

So traditional pts are the closed pts, bonus pts are not closed.
**The Structure Sheaf**

Define $O_{\text{Spec } R}(D(f)) = \text{localization of } R \text{ at the multiplicative set of all functions that do not vanish outside } Z(f)$, i.e. those $g \in R$ s.t. $Z(g) \subseteq Z(f)$ (or $D(f) \subseteq D(g)$).

**Note.** This only depends on $D(f)$, not $f$.

**Fact.** The natural map $Rf \rightarrow O_{\text{Spec } R}(D(f))$ is an exercise.

If $D(f') \subseteq D(f)$ define restriction

$$O_{\text{Spec } R}(D(f)) \rightarrow O_{\text{Spec } R}(D(f'))$$

in the obvious way. The latter ring is a further localization. → pre-sheaf.

**Thm.** This data gives a sheaf. “Affine scheme”

A scheme is a ringed space locally isomorphic to an affine scheme.
Pf of Thm. Let's check gluability in the case of a finite cover of $\text{Spec}(R)$:

$$\text{Spec}(R) = \text{D}(f_1) \cup \cdots \cup \text{D}(f_n)$$

Say we have elts $a_i/f_i \in R_{f_i}$ that agree on the overlaps $R_{f_i}f_j$.

Let $g_i = f_i^h$, so $\text{D}(f_i) = \text{D}(g_i)$.

$\leadsto a_i/g_i \in R_{g_i}$.

To say $a_i/g_i$ & $a_i/g_j$ agree on the overlap (in $A_{g_i}g_j$) means for some $m_{ij}$ :

$$(g_i, g_j)^{m_{ij}}(g_j a_i - g_i a_j) = 0$$

in $R$. Let $m = \max m_{ij}$, so

$$(g_i, g_j)^m(g_j a_i - g_i a_j) = 0 \quad \forall \ i, j.$$ 

Let $b_i = a_i g_i^m \forall i$.

$hi = g_i^{m+1}$ so $\text{D}(h_i) = \text{D}(g_i)$

So: on each $\text{D}(h_i)$ we have a function $b_i/h_i$ and the overlap condition is

$$h_j b_i = h_i b_j$$

Have $\cup \text{D}(f_i) = \text{Spec } R \Rightarrow 1 = \Sigma r_i h_i$ some $r_i \in R$.

Define $r = \Sigma r_i b_i$.

This restricts to each $b_i/h_j$. Indeed

$$r h_j = \Sigma r_i b_i h_j = \Sigma r_i h_i b_j = b_j$$
**Nullstellensatz**

\[ 
\mathbb{II}(S) = \text{fns vanishing on } S. 
\]

**Nullstellensatz:**

\[
\left\{ \text{closed subsets of } \text{Spec}(R) \right\} \leftrightarrow \left\{ \text{radical ideals of } R \right\} \\
X \mapsto \mathbb{II}(X) \\
\mathbb{Z}(I) \leftrightarrow I \\
\left\{ \text{irred. closeds of } \text{Spec}(R) \right\} \leftrightarrow \left\{ \text{prime ideals of } R \right\}
\]
**Visualizing Nilpotents**

**Motivation:** \( \text{Spec } \frac{\mathbb{C}[x]}{(x(x-1)(x-2))} \leftrightarrow \{0,1,2\} \)

The map \( \mathbb{C}[x] \rightarrow \frac{\mathbb{C}[x]}{(x(x-1)(x-2))} \) can be interpreted (via Chinese R.T.) as: take a function on \( \mathbb{A}^1 \), restrict it to \( 0,1,2 \)

What about non-radical ideals?

Consider \( \text{Spec } \frac{\mathbb{C}[x]}{(x^2)} \). As a subset of \( \mathbb{A}^1 \) it is just the origin, which we think of as \( \text{Spec } \frac{\mathbb{C}[x]}{(x)} \). Now want to remember the \( x^2 \).

Image of \( f(x) \) is \( f(0) \) and \( f'(0) \)
Aside: CRT

CRT: Knowing \( n \mod 60 \) is same as knowing \( n \mod 2,3,5 \)

What is Spec \( \mathbb{Z}/(60) \)? The ideals \( (2), (3), (5) \) with discrete top.

The stalks are \( \mathbb{Z}/4, \mathbb{Z}/3, \mathbb{Z}/5 \)
INTERSECTION MULTIPLICITY

For Bézout’s thm, need a notion of intersection multiplicity:

Let \( I \subseteq k[x_0, \ldots, x_n] \) be a homog ideal with finite projective \( 0 \) locus, \( a \in \mathbb{P}^n \).
Choose an affine patch of \( \mathbb{P}^n \) containing \( a \), and let \( J \) be the corresp. affine ideal.

\[
mult_a (I) = \dim_k O_{\mathbb{A}^n, a} / J O_{\mathbb{A}^n, a}
\]

**Example.** \( X = Z(x_0x_2-x_1^2) \) \( Y = Z(x_2) \)

\[
mult_a (X, Y) = mult_a (x_0x_2-x_1^2, x_2) \\
= \dim_k O_{\mathbb{A}^2, 0} / (x_2-x_1^2, x_2) \\
= \dim_k O_{\mathbb{A}^2, 0} / (x_2, x_2) \\
= \dim_k k[x_1, x_2] / (x_2^2, x_2) \\
= \dim_k k[x_2] / (x_2) \\
= 2.
\]