Schemes

Schemes are the main objects of study in algebraic geometry. The main developments are due to Grothendieck in the 1960's.

The (very) basic idea is this: instead of starting with a space X and obtaining a ring $\mathcal{O}_X(X)$, we start with an arbitrary ring R and create a space $\mathrm{Spec}(R)$.

The ring R might have nilpotent elements. We can use these to record higher order intersections. Consider the intersection $Z(y-x^2) \cap Z(y)$. Normally we eliminate y to obtain $Z(x^2) \subseteq \mathbb{A}^1$ then take radical to get $Z(x)$. With schemes, we leave it as $Z(x^2)$, yielding a nilpotent element that records the second order intersection.

One consequence is that there is a Bézout theorem that holds all the time, not just generically.

Another thing that happens in scheme theory is that we can treat varieties over finite fields using geometric intuition from $\mathbb{C}$. We'll see $\mathrm{Spec}(\mathbb{Z})$ consists of $0 \cup \{\text{primes}\}$. Given an algebraic curve with $\mathbb{Z}$ coefficients, we can reduce mod p, yielding a family of "curves", one for each p. Scheme theory allows us to relate these to each other. (Cf. Weil conjectures).

Example: the Frobenius map

We write any polynomial, say with $\mathbb{Z}$-coeffs:

$$f(x) = x^2 - x + 3$$

What are the roots in k, for various k?
How many roots does it have in $\mathbb{F}_7$?

Let $k = \overline{\mathbb{F}_p}$

$$\mathcal{F}_p : \mathbb{A}^n \longrightarrow \mathbb{A}^n$$
$$x_i \longmapsto x_i^p$$

This is a bijection (why?) but not an isomorphism, since $\mathcal{F}_p^* : k[x_1, \dots, x_n] \longrightarrow k[x_1, \dots, x_n]$ is not surjective (x_1 is not in the image).

Fact. For $m \geq 1$, $\mathbb{F}_{p^m}$ is the unique subfield of k with degree m over $\mathbb{F}_p$.
And it equals the set of fixed pts of $\mathcal{F}_p^m$.

Say now $f(x_1, \dots, x_n)$ is a polynomial with coeffs in $\mathbb{F}_{p^m}$, say

$$f = \sum c_I X^I = c_{i_1 \cdots i_n} X^{i_1} \cdots X^{i_n} \qquad c_I \in \mathbb{F}_{p^m}.$$

If $(a_1, \dots, a_n) \in Z(f) \subseteq \mathbb{A}^n_k$ then

$$0 = \mathcal{F}_p^m\left(\sum c_I a_i^{i_i} \cdots a^{i_n}\right) = \sum \mathcal{F}_p^m(a_1)^{i_1} \cdots \mathcal{F}_p^m(a_n)^{i_n}$$

$$\Longrightarrow \mathcal{F}_p^m(a_1, \dots, a_m) \subseteq Z(f).$$

So $\mathcal{F}_p^m$ maps $Z(f)$ to itself. And the fixed points are the points in $\mathbb{F}_{p^m}$.

We wanted to count these. In algebraic topology we would use the Lefschetz fixed point theorem.
How can we do that here? $Z(f)$ is a discrete set!
Answer: define schemes (rings with a topology on their set of prime ideals...), then define étale cohomology for schemes, then come up with a Lefschetz fixed point theorem, solve the Weil conjectures, win the Fields medal...

Spec

Following Vakil

R = commutative ring

Def. The prime spectrum, or spectrum, of R is the collection of prime ideals, denoted $\mathrm{Spec}(R)$.

We also refer to $\mathrm{Spec}(R)$ as the affine scheme associated to R.

Before, points of X corresponded to max. ideals in $k[X]$. So $\mathrm{Spec}(R)$ has "extra" points. Some of these "points" are contained in others.

Note. • R itself is not prime.
• 0 is prime iff R is a domain.

We should think of:
$$\mathrm{Spec}(R) \longleftrightarrow X$$
$$R \longleftrightarrow k[X]$$

Because of this, we'll want to think of elements of R as functions on $\mathrm{Spec}(R)$

To this end: For each $p \in \mathrm{Spec}(R)$ let $k(p)$ denote the quotient field of R/p.

Here is how $f \in R$ can be thought of as a function on $\mathrm{Spec}(R)$: for $p \in \mathrm{Spec}(R)$, let $f(p)$ be the image of f under

$$R \to R/p \longrightarrow k(p)$$

We call $f(p)$ the value of f at p.

Note that these values lie in different fields: the function 5 on $\mathrm{Spec}\,\mathbb{Z}$ takes the value

$$\begin{array}{llll} & 1 \bmod 2 & \text{at} & (2) \in \mathrm{Spec}\,\mathbb{Z} \\ \text{and} & 2 \bmod 3 & \text{at} & (3) \end{array}$$

The statement $f \in p$ translates to $f(p) = 0$.

The fact that we can add & multiply functions pointwise translates to the fact that $R \to k(p)$ is a ring homom.

Will eventually interpret these functions as global sections of the structure sheaf on $\mathrm{Spec}(R)$.

If $R = k[X] = k[x_1,\dots,x_n]/\mathbb{I}(X)$
& p is a max. ideal of R (ie a pt of X).
then $k(p) = k$ and the value of $f \in R$ is the value in the classical sense.

Example. $\mathrm{Spec}(\mathbb{C}[x]) = 0 \cup \{x - a : a \in \mathbb{C}\}$

This is the full set of prime ideals since every polynomial factors into linears.

Let's call this space $\mathbb{A}^1_{\mathbb{C}}$

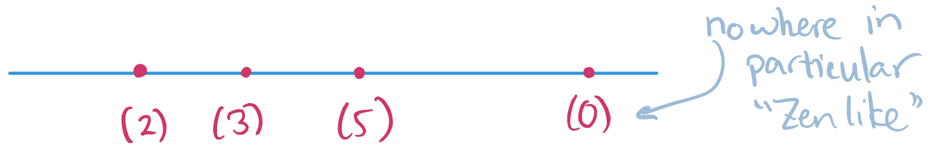
Picture:

The functions on $\mathbb{A}^1_{\mathbb{C}}$ are polynomials
So $f(x) = x^2 - 3x + 1$ is a function. Its value at $(x-1)$ (which we think of as 1) is... $f(1)$. Really we should take the equiv. class of $f(x)$ in $\mathbb{C}[x]/(x-1)$, but this is the same as setting $x = 1$. (by the divison alg)
The value of f at (0) is just $f(x)$. ↳ try it!

This whole discussion works over any alg. closed k.

Example. $\operatorname{Spec}\mathbb{Z} = 0 \cup \{\text{primes}\}$

Same picture:



100 is a function. Its value at 3 is 1.
It has a (double!) zero at 2...

Example. $\operatorname{Spec} k = pt$.

Example. $R = k[\varepsilon]/\varepsilon^2$ "ring of dual numbers" k alg. closed

Think of ε as a small number (its square is 0).

Will show: $\operatorname{Spec}(R) = \{(\varepsilon)\}$.

Indeed: Primes of $R \longleftrightarrow$ primes $p \subseteq k[x]$, $p \supseteq (\varepsilon^2)$

$k[\varepsilon]$ principal so

$p = (f)$ and $(\varepsilon^2) \subseteq f \iff f \mid \varepsilon^2$ $\square$

The function ε is nonzero but its value at all points of $\operatorname{Spec}(R)$ is 0. So:

$\boxed{\text{functions are not determined by their values}}$

It boils down to the fact that the intersection of all prime ideals is not 0.

Example. $\mathrm{Spec}(\mathbb{R}[x]) = \{(0)\} \cup \{(x-a)\} \cup \{\text{irred. quadratics}\}$

Call it $\mathbb{A}^1_{\mathbb{R}}$

The first two pieces are familiar. The new pts are complex conjugate pairs.

div. alg.

Consider $f(x) = x^3 - 1$.

Its value at $(x-2)$ is 7, or $7 \bmod (x-2)$; this $f(2)$.

Its value at (x^2+1) is $-x-1 \bmod (x^2+1)$, which we can think of as $-i-1$.

Example. $\mathbb{A}^1_{\mathbb{F}_p} = \mathrm{Spec}\, \mathbb{F}_p[x] = \{(0)\} \cup \{\text{irred. polys}\}$ (since $\mathbb{F}_p[x]$ is a domain.

Can identify each irred poly with the corresponding set of Galois conjugates in $\overline{\mathbb{F}_p}$.

A polynomial f is not determined by its values on $\mathbb{F}_p$ but is det. by values on $\overline{\mathbb{F}_p}$.

e.g. $f(x) = 1$ & $g(x) = x^2 + x + 1$ with $p = 2$.

Example. $\mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[x,y]$

Note: $\mathbb{C}[x,y]$ not principal: (x,y) is not princ.

$(0) \in \mathbb{A}^2_{\mathbb{C}}$

$(x-a, y-b) \in \mathbb{A}^2_{\mathbb{C}}$ (these are max. ideals)

Other irreducibles also lie in $\mathbb{A}^2_{\mathbb{C}}$, such as

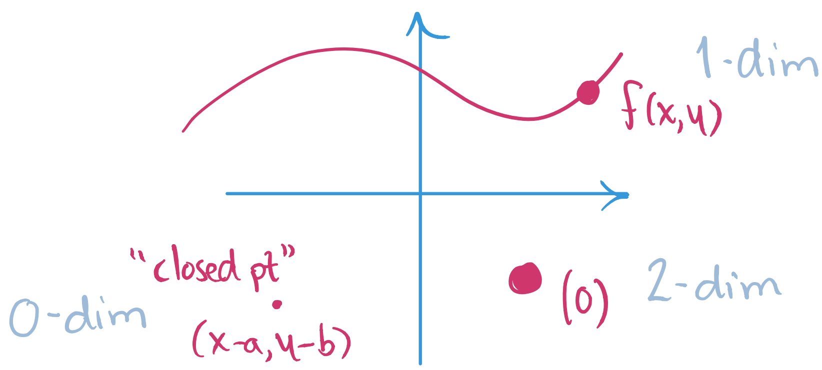
$$y - x^2 \quad \& \quad y^2 - x^3.$$

To picture this, the $(x-a, y-b)$ correspond to points of $\mathbb{C}^2$.

What about the "bonus" points?

(0) is "behind" all the traditional pts. It does not lie on $y = x^2$.

$(y-x^2)$ lies on $y = x^2$, but nowhere partic. on it.



Example $\mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y,z]$

Again there are points of dim...

0 : $(x-a, y-b, z-c)$

3 : (0)

2 : (f) f irred.

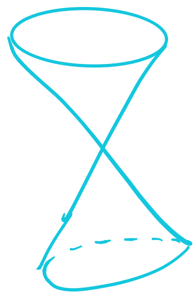
1 : impossible to classify; irred. curves

example: $(x,y) \longleftrightarrow$ z-axis

From Ring Operations to Spec Operations

Quotients. $\operatorname{Spec} R/I \subseteq \operatorname{Spec} R$

Special case: Say R is a fin. gen. $\mathbb{C}$-alg. gen by $x_1, \ldots, x_n$ with relations $f_i(x_1, \ldots, x_n) = 0$.
so $R = k[x_1, \ldots, x_n]/(f_i)$ and $\operatorname{Spec} R/I$ is the set of pts of $\operatorname{Spec} R$ satisfying the f_i,
e.g.



Localizations. $\operatorname{Spec} S^{-1}R \subseteq \operatorname{Spec} R$

Exercise: $\operatorname{Spec} S^{-1}R \longleftrightarrow$ primes in $\operatorname{Spec} R$ not meeting S.

Example. $S = \{1, f, f^2, \ldots\} \subseteq R$
$\rightsquigarrow \operatorname{Spec} S^{-1}R = \{p \in \operatorname{Spec} R : f \notin p\}$

More specific. $R = \mathbb{C}[x,y]$ $f(x,y) = y - x^2$
$\rightsquigarrow \operatorname{Spec} S^{-1}R = \{x \in \operatorname{Spec} R : f \text{ doesn't vanish}\}$
$= \mathbb{A}^2_{\mathbb{C}}$ minus pts on $y - x^2$
and the bonus pt $(y - x^2)$.

Maps. $f: B \to A$ map of rings
$\rightsquigarrow \operatorname{Spec} A \to \operatorname{Spec} B$

Explicit example. $P = \{(a,b) : b = a^2\} \subseteq \mathbb{C}^2$
$C = \{(x,y,z) : z = y^2, y = x^2\} \subseteq \mathbb{C}^3$

Say $f : P \to C$
$(a,b) \mapsto (a,b,b^2)$

$$\rightsquigarrow \operatorname{Spec} \mathbb{C}[a,b]/(b-a^2) \longrightarrow \operatorname{Spec} \mathbb{C}[x,y,z]/(z-y^2, y-x^2)$$

$$\mathbb{C}[a,b]/(b-a^2) \longleftarrow \mathbb{C}[x,y,z]/(z-y^2, y-x^2)$$

$$(a,b,b^2) \longmapsfrom (x,y,z)$$

Nilradicals. The set of nilpotents form an ideal called the nilradical.

Thm. The nilradical is the intersection of all the primes.
Geometrically: a function on $\operatorname{Spec} R$ vanishes everywhere iff it is nilpotent.

Topology

R = comm. ring
$S \subseteq R$ subset

$\rightsquigarrow Z(S) = \{ p \in \operatorname{Spec} R : f(p) = 0 \ \forall f \in S \}$

As usual, the closed sets in Spec R are defined to be the $Z(S)$'s. This is the Zariski topology.

By the definition of value, we also have:

$$Z(S) = \{ p \in \operatorname{Spec}(R) : f \in p \ \forall f \in S \}$$
$$= \{ p \in \operatorname{Spec}(R) : p \supset S \}$$

As usual: $Z(S) = Z((S))$ & $S \subseteq T \Rightarrow Z(T) \subseteq Z(S)$

Example. $Z(xy, yz) \subseteq \mathbb{A}^3_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y,z]$
This is the set of pts with $y=0$ or with $x=z=0$. Also, the bonus points: the generic point of the xz-plane, aka (y) and the gen. pt of y-axis, aka (x,z)
Also: 1-dim pts in xz-plane.

The $Z(S)$ are the closeds for a topology on $\mathrm{Spec}(R)$ since:

(i) $\bigcap Z(I_i) = Z(\sum I_i)$

(ii) $Z(I) \cup Z(J) = Z(IJ)$

(iii) $Z(I) \subseteq Z(J) \Leftrightarrow \sqrt{J} \subseteq \sqrt{I}$

Example. $\mathbb{A}^1_{\mathbb{C}}$

The open sets are: $\emptyset$, $\mathbb{A}^1_{\mathbb{C}}$ minus a finite set of max ideals

Indeed, given $f \in \mathbb{C}[x]$, we factor it

$$f = \prod (x - a_i)$$

So $f \in p_i$ where $p_i = (x - a_i)$. Also, $f \in (0) \Leftrightarrow f = 0$ and f contained in no prime ideals $\Leftrightarrow f = \text{const}$.

So: open sets are determined by their intersections with the traditional pts.

Example. $\mathrm{Spec}\,\mathbb{Z}$

The open sets are $\emptyset$ & complement of finitely many ordinary primes.

Example. $\mathbb{A}^2_{\mathbb{C}}$

Recall the pts are:

max ideals $(x-a, y-b)$	0-dim
$(f(x,y))$ irred.	1-dim
(0)	2-dim

The closed sets are:

- the whole space = closure of (0)

  f vanishes on $(0) \Rightarrow f=0$

- a finite (possibly empty) set of curves (each the closure of a 1-dim pt) and finite number of 0-dim pts

To prove this, the hint is: if $f(x,y)$ and $g(x,y)$ are irred. poly's that are not multiples of each other their 0-sets intersect in a finite # of pts (this follows from fact that $\dim \mathbb{A}^2_{\mathbb{C}} = 2$, proved a long time ago).

Fact. $f: B \to A$

$\rightsquigarrow f^*: \operatorname{Spec} A \to \operatorname{Spec} B$ continuous

i.e. Spec is a contravariant functor Rings $\to$ Top.

Basis for the topology: for $f \in R$,

$$D(f) = \{p \in \mathrm{Spec}(R) : f(p) \neq 0\}$$

Vakil: "Doesn't-vanish set"

Fact. $D(f) \subseteq D(g) \iff f^n \in (g)$ some $n \geq 1$

$\iff g$ invertible in A_f

Pf idea. $Z(g) \longleftrightarrow \mathrm{Spec}(R/(g))$

$D(g) = Z(g)^c$

$\Rightarrow f =$ zero function on $Z(g) = \mathrm{Spec}\, R/(g)$

$\Rightarrow f$ nilpotent on $R/(g)$

i.e. $f^n \in (g)$.

Def. In a top. space, we say a point is

- closed if it is its own closure
- generic if its closure is the whole space.
- generic in a closed set K if its closure is K.

We say x is a specialization of y if $x \in \overline{\{y\}}$

eg $(x-7, y-49)$ is a specialization of $(y-x^2)$.

Fact. The closed pts of $\mathrm{Spec}\, R$ are the max ideals.

So traditional pts are the closed pts, bonus pts are not closed.

The Structure Sheaf

Define $\mathcal{O}_{\operatorname{Spec} R}(D(f))$ = localization of R at the multiplicative set of all functions that do not vanish outside $Z(f)$, i.e. those $g \in R$ s.t. $Z(g) \subseteq Z(f)$ (or $D(f) \subseteq D(g)$).

Note. This only depends on $D(f)$, not f.

Fact. The natural map $R_f \longrightarrow \mathcal{O}_{\operatorname{Spec} R}(D(f))$

exercise.

If $D(f') \subseteq D(f)$ define restriction

$$\mathcal{O}_{\operatorname{Spec} R}(D(f)) \longrightarrow \mathcal{O}_{\operatorname{Spec} R}(D(f'))$$

in the obvious way. The latter ring is a further localization. $\rightsquigarrow$ pre-sheaf.

Thm. This data gives a sheaf. "Affine scheme"

A scheme is a ringed space locally isomorphic to an affine scheme.

Pf of Thm. Let's check gluability in the case of a finite cover of Spec(R):

$$\operatorname{Spec}(R) = D(f_1) \cup \cdots \cup D(f_n)$$

Say we have elts $a_i/f_i^{l_i} \in R_{f_i}$ that agree on the overlaps $R_{f_i f_j}$

Let $g_i = f_i^{l_i}$, so $D(f_i) = D(g_i)$.

$\rightsquigarrow$ $a_i/g_i \in R_{g_i}$.

To say a_i/g_i & a_j/g_j agree on the overlap (in $A_{g_i g_j}$) means for some m_{ij}:

$$(g_i g_j)^{m_{ij}}(g_j a_i - g_i a_j) = 0.$$

in R. Let $m = \max m_{ij}$, so

$$(g_i g_j)^m (g_j a_i - g_i a_j) = 0 \quad \forall\, i,j.$$

Let $b_i = a_i g_i^m \ \forall i$.

$h_i = g_i^{m+1}$ so $D(h_i) = D(g_i)$

Basically the same proof as before!

So: on each $D(h_i)$ we have a function b_i/h_i and the overlap condition is

$$h_j b_i = h_i b_j$$

Have $\bigcup D(f_i) = \operatorname{Spec} R \Rightarrow 1 = \sum r_i h_i$ some $r_i \in R$.

Define $r = \sum r_i b_i$.

This restricts to each b_j/h_j. Indeed

$$r h_j = \sum r_i b_i h_j = \sum r_i h_i b_j = b_j \qquad \square$$

Nullstullensatz

$\mathbb{I}(S)$ = fns vanishing on S.

Nullstullensatz:

$$\left\{\begin{array}{c}\text{closed subsets} \\ \text{of } \operatorname{Spec}(R)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{radical ideals} \\ \text{of } R\end{array}\right\}$$

$$X \longmapsto \mathbb{I}(X)$$

$$Z(I) \longleftarrow I$$

$$\left\{\begin{array}{c}\text{irred. closeds} \\ \text{of } \operatorname{Spec}(R)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{prime ideals} \\ \text{of } R\end{array}\right\}$$

Visualizing Nilpotents

Motivation: $\text{Spec } \mathbb{C}[x]/(x(x-1)(x-2)) \longleftrightarrow \{0,1,2\}$

The map $\mathbb{C}[x] \longrightarrow \mathbb{C}[x]/(x(x-1)(x-2))$ can be interpreted (via Chinese R.T.) as: take a function on $\mathbb{A}^1$, restrict it to $0,1,2$

What about non-radical ideals?

Consider $\text{Spec } \mathbb{C}[x]/(x^2)$. As a subset of $\mathbb{A}^1$ it is just the origin, which we think of as $\text{Spec } \mathbb{C}[x]/(x)$. Now want to remember the x^2.

Image of $f(x)$ is $f(0)$ and $f'(0)$

Aside: CRT

CRT: Knowing n mod 60 is same as knowing
n mod 2,3,5

What is Spec $\mathbb{Z}/(60)$? The ideals $(2), (3), (5)$.
with discrete top.
The stalks are $\mathbb{Z}/4$, $\mathbb{Z}/3$, $\mathbb{Z}/5$

Intersection Multiplicity

For Bézout's thm, need a notion of intersection multiplicity:

Let $I \subseteq k[x_0, \dots, x_n]$ be a homog. ideal with finite projective 0 locus, $a \in \mathbb{P}^n$.
Choose an affine patch of $\mathbb{P}^n$ containing a, and let J be the corresp. affine ideal.

$$\mathrm{mult}_a(I) = \dim_k \mathcal{O}_{\mathbb{A}^n, a} / J\mathcal{O}_{\mathbb{A}^n, a}$$

Example. $X = Z(x_0x_2 - x_1^2) \quad Y = Z(x_2)$

$$\begin{aligned}
\mathrm{mult}_a(X,Y) &= \mathrm{mult}_a(x_0x_2 - x_1^2, x_2) \\
&= \dim_k \mathcal{O}_{\mathbb{A}^2, 0} / (x_2 - x_1^2, x_2) \\
&= \dim_k \mathcal{O}_{\mathbb{A}^2, 0} / (x_1^2, x_2) \\
&= \dim_k k[x_1, x_2] / (x_1^2, x_2) \\
&= \dim_k k[x_1] / (x_1^2) \\
&= 2.
\end{aligned}$$