The Sheaf of Regular Functions

Want an analogue of $f: M \to \mathbb{R}$ in diff. top
or $f: M \to \mathbb{C}$ in complex analysis.
We defined morphisms on aff. alg. var's. Want a version for open subsets.
A map should be a morphism if it is a morphism on a nbd of each pt.

Def. X = aff. alg. var.
$U \subseteq X$ open.
A regular fn on U is a map $\varphi: U \to k$ with: $\forall\, a \in U\ \exists$ poly. fns f, g with $f(x) \neq 0$ and

$$\varphi(x) = g(x)/f(x)$$

$\forall\, x$ in an open subset U_a with $a \in U_a \subseteq U$
The set of all such regular fns is denoted $\mathcal{O}_X(U)$.

Note. $\mathcal{O}_X(U)$ is a k-algebra

Example. $X = Z(x_1x_4 - x_2x_3) \subseteq \mathbb{A}^4$.

$U = X \setminus Z(x_2, x_4)$

$\varphi : U \longrightarrow k$

$$(x_1, \dots, x_4) \longmapsto \begin{cases} x_1/x_2 & x_2 \neq 0 \\ x_3/x_4 & x_4 \neq 0. \end{cases}$$

This φ is a reg. fn on U.

Indeed it is well-def since $x_1x_4 - x_2x_3 = 0$
$\Rightarrow x_1/x_2 = x_3/x_4$ when $x_2, x_4 \neq 0$.

Also, it is locally a rational fn.

Note: neither formula works at all points, e.g. first formula fails at $(0,0,0,1)$.

In fact there is no global way to write φ.

Fact. X = aff. alg. var. $U \subseteq X$ open. $\varphi \in \mathcal{O}_X(U)$.
Then $Z(\varphi) = \{x \in U : \varphi(x) = 0\}$ is closed in U.

Pf. By defn, any pt $a \in U$ has a nbd $U_a \subseteq U$ where $\varphi = g_a/f_a$ ($f_a \neq 0$ on U_a). So

$Z_a = \{x \in U_a : \varphi(x) \neq 0\} = U_a \setminus Z(g_a)$

is open in $X \Rightarrow \bigcup Z_a$ open in X

But $\bigcup Z_a = U \setminus Z(\varphi)$.

So $Z(\varphi)$ closed. $\square$

As a consequence we have...

Fact (identity thm for regular fns). X = irred aff alg. var. $\emptyset \neq U \subseteq V$ open. If $\varphi_1, \varphi_2 \in \mathcal{O}_X(V)$ agree on U then they agree on V.

Pf. The set $Z(\varphi_1 - \varphi_2)$ contains U and is closed by the previous fact. Hence it contains $\bar{U}$ = closure of U in V.

But $\bar{V} = X$ (open subsets of irred spaces are dense).

Thus V is irred (in a top. space A irred $\Leftrightarrow \bar{A}$ irred).

$\Rightarrow \bar{U} = V$ $\square$

The fact should not be surprising since open sets in the Zariski top. are big (dense in Euc. top.). So for $k = \mathbb{C}$ the fact is obvious.

The interesting thing is the analogy with the identity theorem in complex analysis: if two holom. fns agree on an open set, they are equal.

Next goal: compute $\mathcal{O}_X(U)$ explicitly in some cases.

Def. X = affine alg. var. $\subseteq \mathbb{A}^n$

$f \in k[X]$ ← usually think of as set of fns on X, not as quotient

$D(f) = X \setminus Z(f)$ is called the *distinguished open subset* of f in X.

Fact ① $D(f) \cap D(g) = D(fg)$

$\rightsquigarrow$ finite intersections of $D(f)$'s are $D(f)$'s.

② Any open subset of X is a finite union of $D(f)$'s:

$$U = X \setminus Z(f_1, \dots, f_k) = D(f_1) \cup \dots \cup D(f_k).$$

So $D(f)$'s are the "smallest" open subsets.

(they form a basis for the Zariski top.)

Prop. X = affine alg. var., $f \in k[X]$. Then

$$\mathcal{O}_X(D(f)) = \left\{ g/f^n : g \in k[X],\ n \in \mathbb{N} \right\}.$$

In partic, setting $f = 1$, $\mathcal{O}_X(X) = k[X]$.

Pf. $\boxed{\supseteq}$ Obvious. Each g/f^n is regular on $D(f)$.

$\boxed{\subseteq}$ Let $\varphi \in \mathcal{O}_X(D(f))$.

By defn: $\forall a \in D(f)$ $\varphi = g_a/f_a$ on nbd of a

After possibly shrinking the nbd's, can assume they are all of the form $D(h_a)$ (above fact).

Claim: We can assume $h_a = f_a$.

and that g_a vanishes on $Z(h_a)$

Pf of Claim: Can rewrite φ locally as $g_a h_a / f_a h_a$,

Note $f_a h_a$ and $g_a h_a$ vanish on each $Z(h_a)$

Also, $f_a h_a$ does not vanish on $D(h_a)$.

But h_a also vanishes exactly on $Z(h_a)$

So $f_a h_a$ & h_a have same zero set. Replace h_a w/ $f_a h_a$.

From now on we make the assumption from the claim.

Claim: $g_a f_b = g_b f_a \ \forall a, b \in D(f)$.

Pf of Claim: These fns agree on $D(f_a) \cap D(f_b)$ since $\varphi = g_a / f_a = g_b / f_b$ there. And they are zero otherwise (need to use 2^{nd} statement of previous claim).

The $D(f_a)$ cover $D(f)$. Pass to the complement:

$$Z(f) = \bigcap_{a \in D(f)} Z(f_a) = Z(\{f_a : a \in D(f)\})$$

So:

$$I(Z(f)) = I(Z(\{f_a\})) = \sqrt{\{f_a : a \in D(f)\}}$$

$$f \in I(Z(f)) \Longrightarrow f \in \sqrt{\{f_a : a \in D(f)\}}$$

$\Rightarrow f^n = \sum_a k_a f_a$ some $n \in \mathbb{N}$, $k_a \in k[X]$ for finitely many $a \in D(f)$.

Let $g = \sum k_a g_a$.

To finish the proof, will show $\varphi = g/f^n$ on $D(f)$:

$\forall\ b \in D(f)$ have $\varphi = g_b/f_b$ and

$$g f_b = \sum_a k_a g_a f_b = \sum_a k_a g_b f_a = g_b f^n$$

on $D(f_b)$.

Thus $\varphi = g_b/f_b = g/f^n$ (both denoms nonzero). The open subsets cover $D(f)$, so done. $\square$

Note. Really need k alg. closed.
For example $1/x^2+1$ is regular on $\mathbb{A}^1_{\mathbb{R}}$ but not in $\mathbb{R}[x]$.

There is an algebraic interpretation of what we just did...

LOCALIZATIONS

R = ring.
S = mult. closed subset (so $1 \in S$).
The localization of R at S is

$$S^{-1}R = \{f/g : f \in R, g \in S\}/\sim$$

not needed in an integral dom.

where $f/g \sim f'/g'$ if $\exists\, h \in S$ s.t. $h(fg' - f'g) = 0$.

Later: germs of fns $\longleftrightarrow$ localizations

When $S = \{f^n : n \in \mathbb{N}\}$ write R_f for $S^{-1}R$.

Lemma. X = aff. alg. var.
$f \in k[X]$
Then $\mathcal{O}_X(D(f)) \cong k[X]_f$ (as k-alg's).

Pf. There is a k-alg. map

$$k[X]_f \longrightarrow \mathcal{O}_X(D(f))$$

$$g/f^n \longmapsto g/f^n$$

formal fraction

actual quotient of polynomials

Check this is well defined:

if $g/f^n \sim g'/f^m$ then

$f^k(gf^m - g'f^n) = 0$ in $k[X]$ some $k \in \mathbb{N}$.

$\Rightarrow gf^m = g'f^n \Rightarrow g/f^n = g'/f^m$ as fns.

Surjectivity: The last Prop.

Injectivity: If $g/f^n \equiv 0$ as a fn on $D(f)$

then $g \equiv 0$ on $D(f)$.

$\Rightarrow fg \equiv 0$ on X

$\Rightarrow f(g \cdot 1 - 0 \cdot f^n) = 0$ in $k[X]$

$\Rightarrow g/f^n \sim 0/1$ $\square$

Example: reg. fns on $\mathbb{A}^2 \setminus 0$.

Let $X = \mathbb{A}^2$, $U = \mathbb{A}^2 \setminus 0$.

Will show: $\mathcal{O}_X(U) = k[x_1, x_2]$.

i.e. $\mathcal{O}_X(U) = \mathcal{O}_X(X)$.

This is an analog of the removable singularity thm in complex analysis: a holom. fn on $\mathbb{C} \setminus 0$ can be extended.

Let $\varphi \in \mathcal{O}_X(U)$.

Prop $\Rightarrow$ on the open sets $D(x_1) = (\mathbb{A}^1 \setminus 0) \times \mathbb{A}^1$ and $D(x_2) = \mathbb{A}^1 \times (\mathbb{A}^1 \setminus 0)$ can write φ as f/x_1^m and g/x_2^n with $f, g \in k[x_1, x_2]$.

WLOG $x_1 \nmid f$, $x_2 \nmid g$.

On $D(x_1) \cap D(x_2)$ both representations are valid.

$\rightarrow$ $fx_2^n = gx_1^m$.

But $Z(fx_2^n - gx_1^m)$ is closed

$\Rightarrow$ $fx_2^n = gx_1^m$ on $\overline{D(x_1) \cap D(x_2)} = \mathbb{A}^2$

$\Rightarrow$ $fx_2^n = gx_1^m$ in $k[\mathbb{A}^2] = k[x_1, x_2]$.

If $m > 0$ then $x_1 \nmid f$, contradiction. $\square$

SHEAVES

A presheaf of rings $\mathcal{F}$ on a topological space X consists of data:

- $\forall$ open $U \subseteq X$ a ring $\mathcal{F}(U)$ $\leftarrow$ the ring of fns
- $\forall$ opens $U \subseteq V \subseteq X$ a ring hom

  $$\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$$

  called the restriction map.

Such that

- $\mathcal{F}(\emptyset) = 0$
- $\rho_{U,U} = \mathrm{id}$ $\forall$ U.
- $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$ $\forall$ $U \subseteq V \subseteq W$.

The elts of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.

The $\rho_{V,U}$ are written as $\varphi \mapsto \varphi|_U$.

The presheaf $\mathcal{F}$ is called a sheaf of rings if it satisfies the following gluing property:

if $U \subseteq X$ open, $\{U_i\}_I$ is an open cover of U and $\varphi_i \in \mathcal{F}(U_i)$ sections $\forall i$ s.t. $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ $\forall$ $i,j \in I$ then $\exists!$ $\varphi \in \mathcal{F}(U)$ s.t. $\varphi|_{U_i} = \varphi_i$ $\forall i$.

Examples. ① X = irred aff. alg. var.

The $\mathcal{O}_X(U)$, plus usual restriction, form a sheaf. The presheaf axioms are clear. The gluing property means $\varphi : U \to k$ is regular if it is regular on each set of an open cover. But a fn is reg. iff it is <u>locally</u> rational.

This $\mathcal{O}_X$ is the **sheaf of regular fns** on X

② $X = \mathbb{R}^n$, $\mathcal{F}(U) = \{\varphi : U \to \mathbb{R} \text{ continuous}\}$

Similar for differentiable fns, analytic fns, arbitrary fns...

③ $X = \mathbb{R}^n$, $\mathcal{F}(U) = \{\text{constant fns}\}$

This is a presheaf, but not a sheaf:

Let U_1, U_2 nonempty disjoint open sets, $\varphi_i \in U_i$ const. fns with different values...

"being constant is not a local condition"

Can fix by taking <u>locally</u> const. fns.

Aside. A smooth manifold is a sheaf of $\mathbb{R}$-algebras $\mathcal{O}_M$ on a (second countable, Hausdorff) topological space M that is locally isomorphic to the sheaf of smooth fns on $\mathbb{R}^n$.

Germs. Let $\mathcal{F}$ be a presheaf on a top. sp. X.
Fix $a \in X$ and consider pairs (U, φ) where U is an open nbd of a & $\varphi \in \mathcal{F}(U)$.
Say $(U, \varphi) \sim (U', \varphi')$ if there is an open V with $a \in V \subseteq U \cap U'$ and $\varphi|_V = \varphi'|_V$.
The set of all equiv. classes is the stalk $\mathcal{F}_a$ of $\mathcal{F}$ at a. It inherits a ring structure from the rings $\mathcal{F}(U)$. The elts of $\mathcal{F}_a$ are called the germs of $\mathcal{F}$ at a.

Germs are also referred to as local fns. For smooth fns, can calculate all derivatives from the germ.

If φ_1, φ_2 are regular fns on open subset U of aff alg var X and they represent the same germ at $a \in U$ then they agree on all of U (identity thm).

Next: germs $\longleftrightarrow$ localizations.

Lemma. X = aff. alg. var., $a \in X$

$S = \{f \in k[X] : f(a) \neq 0\}$

The stalk $\mathcal{O}_{X,a}$ of $\mathcal{O}_X$ is k-alg isomorphic to

$$S^{-1}k[X] = \{g/f : f, g \in k[X],\ f(a) \neq 0\}$$

This is called the local ring of X at a.

Pf. Note S is mult. closed, so the lemma makes sense.

Have a k-alg hom:

$$\begin{aligned} S^{-1}k[X] &\longrightarrow \mathcal{O}_{X,a} \\ g/f &\longmapsto \overline{(D(f), g/f)} \end{aligned}$$

equiv. class

Check well def, inj, surj. $\square$

Lemma/Defn X = aff. alg. var., $a \in X$

Every proper ideal of $\mathcal{O}_{X,a}$ is contained in the ideal

$$I_a = I(a)\mathcal{O}_{X,a} = \{g/f \mid f, g \in k[X],\ g(a) = 0,\ f(a) \neq 0\}$$

called the maximal ideal of $\mathcal{O}_{X,a}$.

Pf. I_a is clearly an ideal ✓

Say $I \subseteq \mathcal{O}_{X,a}$ an ideal not contained in I_a.

$\rightsquigarrow \exists\ g/f \in I$ with $f(a), g(a) \neq 0$. $\Rightarrow f/g \in \mathcal{O}_{X,a}$

$\Rightarrow 1 \in \mathcal{O}_{X,a} \Rightarrow I = \mathcal{O}_{X,a}$ $\square$

Morphisms

A ringed space is a top. space X with a sheaf of rings $\mathcal{O}_X$. We call $\mathcal{O}_X$ the structure sheaf.

An affine variety is a ringed space with its sheaf of regular fns.

An open subset of a ringed space is a ringed space (retrict).

Want to say $X \longrightarrow Y$ is a morphism if it pulls elts of $\mathcal{O}_Y(Y)$ to $\mathcal{O}_X(U)$. But elts of $\mathcal{O}_X(U)$ are not necessarily fns. So:

$\boxed{\text{From now on sheaves are sheaves of } k\text{-valued fns}}$

Defn. Let $f: X \longrightarrow Y$ be a map of ringed spaces. Then f is a morphism if it is continuous and if $\forall$ open $U \subseteq Y$ and $\varphi \in \mathcal{O}_Y(U)$ we have $f^*\varphi \in \mathcal{O}_X(f^{-1}(U))$.

So, for a morphism gives k-alg homom's

$$f^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}(U)).$$

Notes.
- Morphisms & isomorphisms of (open subsets of) affine alg. var's are morphism & isomorphisms of ringed spaces.
- Continuity is used so $f^{-1}(U)$ open.
- Compositions of morphisms are morphisms.
- Restrictions of morphisms are morphisms: if $U \subseteq X$ open & $f(U) \subseteq V$ open in Y then $f|_U$ is a morphism.

Morphisms have a gluing property:

Lemma X, Y ringed spaces, $f: X \to Y$
$\{U_i\}$ open cover of X s.t. each $f|_{U_i}$ is a morphism.
Then f is a morphism.

Pf. Continuity: works since continuity is local.
Pullbacks: Let $V \subseteq Y$ open, $\varphi \in \mathcal{O}_V(Y)$
Then $(f^*\varphi)|_{U_i \cap f^{-1}(V)} = (f|_{U_i \cap f^{-1}(V)})^* \varphi$
lies in $\mathcal{O}_X(U_i \cap f^{-1}(V))$ since $f|_{U_i}$ a morphism $\Rightarrow$
$f|_{U_i \cap f^{-1}(V)}$ is a morphism.
Gluing property for sheaves $\to f^*\varphi \in \mathcal{O}_X(f^{-1}(V))$.

Prop. U = open subset of aff. alg. var X
Y = aff. alg. var. $\subseteq \mathbb{A}^n$
The morphisms $f: U \to Y$ are exactly the maps of the form $f = (\varphi_1, \dots, \varphi_n)$ with $\varphi_i \in \mathcal{O}_X(U)$

In particular, the morphisms $U \to \mathbb{A}^1$ are exactly the elts of $\mathcal{O}_X(U)$.

Pf. Assume $f: U \to Y$ a morphism
The coords fns $y_1, \dots, y_n : Y \to k$ are regular
So $\varphi_i = f^* y_i$ lies in $\mathcal{O}_X(f^{-1}(Y)) = \mathcal{O}_X(U)$
Thus f is of the given form.

Now say $f = (\varphi_1, \dots, \varphi_n)$ as above.

Claim. f is continuous.
Pf of claim. Say $Z \subseteq Y$ closed $\rightsquigarrow Z = Z(g_1, \dots, g_m)$ with $g_i \in k[Y]$. And $f^{-1}(Z) = \{x \in U : g_i(\varphi_1(x), \dots, \varphi_n(x)) = 0\}$
But the fns $g_i(\varphi_1(x), \dots, \varphi_n(x))$ are regular on U
(compositions of rational fns w/ poly's are rational)
$\Longrightarrow f^{-1}(Z)$ closed.

Claim. f^* takes regular fns to reg fns

see note at very bottom

Pf of claim. Say $\varphi \in \mathcal{O}_Y(W)$ regular. Then

$$f^*\varphi = \varphi \circ f : f^{-1}(W) \longrightarrow k$$

is regular for the same reason. $\square$

Seems subtle! Take $f = \left(\frac{x+1}{x-1}, \frac{x+2}{x-2}\right)$, $\varphi = y_1/y_2$, $-2 \in U$

Cor. X, Y affine alg. var's

$$\left\{\begin{matrix}\text{morphisms} \\ X \to Y\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}k\text{-alg homoms} \\ k[Y] \to k[X]\end{matrix}\right\}$$

Pf. $\boxed{\longrightarrow}$ already done

$\boxed{\longleftarrow}$ Let $g : k[Y] \to k[X]$ k-alg homom.

Say $Y \subseteq \mathbb{A}^n$, $y_1, \dots, y_n$ the coord fns

Then $\varphi_i = g(y_i)$ lies in $k[X] = \mathcal{O}_X(X)$ $\forall i$.

Set $f = (\varphi_1, \dots, \varphi_n) : X \to \mathbb{A}^n$

Then $f(X) \subset Y = Z(I(Y))$ subtle!

Prop $\Rightarrow f$ a morphism.

By construction $f^* = g$ $\square$

Example. A bijective morphism that's not an isomorphism.

$X = Z(x_1^2 - x_2^3)$

$f : \mathbb{A}^1 \to X \qquad t \longmapsto (t^3, t^2)$

$\rightsquigarrow f^* : k[x_1, x_2]/(x_1^2 - x_2^3) \longrightarrow k[t] = k[\mathbb{A}^1]$

$\bar{x}_1 \longmapsto t^3, \quad \bar{x}_2 \longmapsto t^2$

Not an $\cong$ since t not in image.

Morphisms and Products

$X = Z(I) \subseteq \mathbb{A}^n$, $Y = Z(J) \subseteq \mathbb{A}^m$ aff. alg. var's
$\rightsquigarrow X \times Y = Z(I, J) \subseteq \mathbb{A}^n \times \mathbb{A}^m$

Note $X \times Y$ does not have product topology. For instance $\Delta = \{(x,x)\}$ is closed in $\mathbb{A}^1 \times \mathbb{A}^1$ but not in the product topology (exercise)

Prop (Univ. prop. for products). X, Y = aff. alg. var's. π_X, π_Y proj's of $X \times Y$ to factors. Then for any aff alg. var Z and morphisms $f_X : Z \to X$, $f_Y : Z \to Y$ there is a unique morphism $f : Z \longrightarrow X \times Y$ s.t. $f_X = \pi_X \circ f$, $f_Y = \pi_Y \circ f_Y$

So: giving a morphism to $X \times Y$ is same as giving a morphism to each factor.

Pf. Uniqueness obvious: only one choice for f. This is a morphism by the last Prop, which characterizes morphisms. $\square$

The univ. prop. for $X \times Y$ corresponds to univ prop. for tensor prod. of coord rings $\Longrightarrow k[X \times Y] \cong k[X] \otimes k[Y]$

Affine Varieties

Recall we showed:

$$\left\{\begin{matrix}\text{affine alg} \\ \text{vars}\end{matrix}\right\}/\sim \quad \longleftrightarrow \quad \left\{\begin{matrix}\text{fin. gen.} \\ k\text{-alg's}\end{matrix}\right\}/\sim$$

The construction of a variety from a presentation of a k-alg can give different aff. alg. var's, depending on the presentation. The above Cor implies the var's are isomorphic. So...

> From now on, an affine variety is a ringed space isomorphic to an aff. alg var, that is, a $Z(I)$.

Prop. X = aff. var, $f \in k[X]$. In the proof we take X to be an aff. alg var $Z(I)$.

$\Longrightarrow D(f)$ is an affine variety with

$$k[D(f)] = k[X]_f \longleftarrow \text{localization.}$$

Pf. Have $Y = \{(x,t) \in X \times \mathbb{A}^1 : t f(x) = 1\} \subseteq X \times \mathbb{A}^1$ is an aff. alg var, since $Y = Z(t f(x) - 1)$.

This Y is isomorphic to $D(f)$ via

$$f: Y \to X \qquad\qquad f^{-1}: X \to Y$$

$$(x,t) \longmapsto x. \qquad\qquad x \longmapsto \left(x, \frac{1}{f(x)}\right)$$

$\Longrightarrow X \cong Y$. We already showed $\mathcal{O}_X(D(f)) \cong k[X]_f$ and $\mathcal{O}_X(X) = k[X]$. $\square$

Example. $\mathbb{A}^2 - \{0\}$ is not an aff. var.

Let $X = \mathbb{A}^2$ and $U = \mathbb{A}^2 - \{0\}$.

Give U the sheaf structure $\mathcal{O}_U(U) = \mathcal{O}_X(U)$

If $\mathcal{O}_U(U)$ were an aff. var we would have

$$\mathcal{O}_U(U) \cong k[U].$$

But we already showed for this X, U that

$$\mathcal{O}_X(U) = \mathcal{O}_X(X) \cong k[x,y] \cong k[\mathbb{A}^2] \text{ (remov. sing. thm)}$$

Inclusion $U \hookrightarrow X$ induces $id: k[x,y] \longrightarrow k[x,y]$

In our proof of the correspondence b/w k-alg homoms & variety homoms, the id. map $k[x,y] \circlearrowleft$ must be induced by inclusion $(\varphi_i = g(y_i) = y_i)$.

This is a contradiction, since $U \hookrightarrow X$ is not surjective. $\square$

We can cover U by $D(x_1)$ & $D(x_2)$, which are affine. So maybe we should allow ringed spaces that are covered by aff. vars...

About that claim further up: perhaps we should thing of a regular fn as a rational fn that is well defined (instead of a rational fn where the denominator does not vanish). Then the claim is easy: the composition of two rational fns is rational, and the composition of two well-defined fns is well def. In particular, the composition of two regular fns is regular.