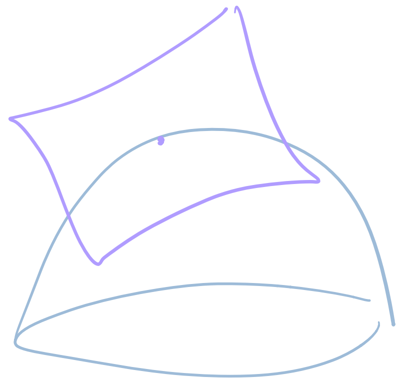
Smoothness

Tangent Space at a point

Idea of tangent space:



Will give several equivalent ways of defining the tangent space to an affine alg. variety.

$\boxed{\text{Method 1: Double roots}}$

Consider $f(x) = x^2$. The graph $y = f(x)$ is tangent to $y=0$. This corresponds to the fact that x^2 has a double root.

More generally, let $V = Z(f_1, \dots, f_r) \subseteq \mathbb{A}^n$
let $p \in V$. WLOG $p = 0$.
Let ℓ be a line thru 0 and $q = (a_1, \dots, a_n)$.
So $\ell = \{(ta_1, \dots, ta_n) : t \in k\}$

When is ℓ tangent to V at p?

Have: $V \cap \ell$ given by solving for t in

$$\left.\begin{aligned} f_1(ta_1,\dots,ta_n) &= 0 \\ &\vdots \\ f_r(ta_1,\dots,ta_n) &= 0. \end{aligned}\right\} \text{poly's in } t$$

By assumption $t=0$ is a soln.

The multiplicity of $V \cap \ell$ at 0 is the highest power of t dividing each $f_i(tq)$.

Def. ℓ is tangent to V at p if the multiplicity of $V \cap \ell$ at p exceeds 1.

The tangent space T_pV is union of the tangent lines.

Two things to check: ① T_pV is indep. of choice of f_i's

② T_pV is a linear subspace.

Examples. ① $V = Z(x^2 - y) \subseteq \mathbb{A}^2$

Say $\ell = \{(ta, tb)\}$

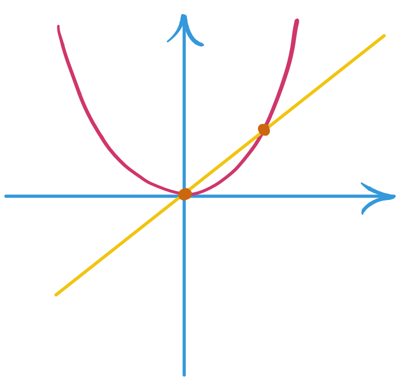
$\rightsquigarrow$ $t^2a^2 - tb = 0$

$\rightsquigarrow$ $t = 0, \ b/a^2$

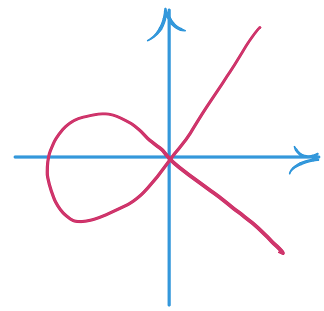
$\rightsquigarrow$ intersection pts $(0,0)$ $\left(b/a, (b/a)^2\right)$

So ℓ tangent $\iff b = 0$

So $T_0V = \{x\text{-axis}\}$



② $V = Z(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2$



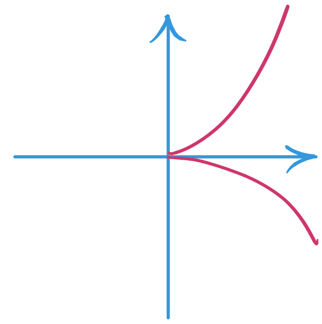
Say $\ell = \{(ta, tb)\}$

$\rightsquigarrow t^2b^2 - t^2a^2 - t^3a^3 = 0$

For all values of a, b we have that t is a multiple root

$\Rightarrow$ all lines tangent, $T_0V = \mathbb{A}^2$

③ $V = Z(y^2 - x^3)$



$\rightsquigarrow t^2b^2 - t^3a^3$

Ditto.

Method 2: Derivatives

The differential of $f \in k[x_1, \dots, x_n]$ at p is the linear part of the Taylor series expansion of f at p. That is, if we write f as

$$f(x) = f(p) + L(x_1 - p_1, \dots, x_n - p_n) + G(x_1 - p_1, \dots, x_n - p_n)$$

where L is linear & G has no linear or const. terms the differential of f at p is $L(x-p)$.

In symbols:

$$L(x-p) = df|_p(x-p) = \sum_{j=1}^{n} \frac{df}{dx_j}(x_j - p_j)$$

Thm. $V = Z(f_1, \dots, f_r) \subseteq \mathbb{A}^n$

Assume $I(V) = (f_1, \dots, f_r)$ i.e. $(f_1, \dots, f_r)$ radical.

Let $p \in V$. Then

① $T_pV = Z(df_1|_p(x-p), \dots, df_r|_p(x-p)) \subseteq \mathbb{A}^n$.

② Moreover T_pV is indep of choice of f_i.

Pf. WLOG $p = 0$.

Say $\ell = \{(tx_1, \dots, tx_n)\}$

Since $p = 0 \in V$, $f_i(0) = 0 \ \forall i$.

$$\rightsquigarrow f_i(tx_1, \dots, tx_n) = L_i(tx_1, \dots, tx_n) + G_i(tx_1, \dots, tx_n)$$
$$= t L_i(x_1, \dots, x_n) + t^2 G_i'(x_1, \dots, x_n, t).$$

So $\ell \subseteq T_0V \iff L_i = 0 \ \forall i$, whence ①.

For ② say $V = Z(g_1, \dots, g_s)$

$\rightsquigarrow f_i = h_{i1} g_1 + \dots + h_{is} g_s$

$\rightsquigarrow df_i = dh_{i1} g_1 + \dots + dh_{is} g_s + h_{i1} dg_1 + \dots + h_{is} dg_s$

Since $g_i(p) = 0$ have

$$df_i|_p = h_{i1} dg_1|_p + \dots + h_{is} dg_s|_p$$

$\implies Z(df_1|_p, \dots, df_r|_p) \supset Z(dg_1|_p, \dots, dg_s|_p)$

and vice versa. $\square$

Note: ① $\implies$ T_pV linear.

Examples. ① $V = Z(x^2-y)$ at $p=0$.

$$\rightsquigarrow \; 2x|_0 \cdot x - 1|_0 \cdot y = 0$$

$$0 \cdot x - 1 \cdot y = 0$$

$$y = 0 \quad \checkmark$$

② $V = Z(y^2-x^2-x^3) \subseteq \mathbb{A}^2$

$$\rightsquigarrow \; 2x-3x^2|_0 \cdot x + 2y|_0 \cdot y = 0$$

$$0 = 0 \quad \checkmark$$

③ $V = Z(y^2-x^3)$ similar when $p=0$.

For $p=(1,1)$:

$$-3x^2|_{(1,1)} \cdot (x-1) + 2y|_{(1,1)} \cdot (y-1) = 0$$

$$-3(x-1) + 2(y-1) = 0.$$

④ $V = Z(x^m - y^m)$

$\rightsquigarrow$ $T_{(a,b)}V$ given by

$$ma^{m-1}(x-a) + mb^{m-1}(y-b) = 0.$$

Assuming $\operatorname{char} k \nmid m$, this is a line.

Note. Can define T_pV for V (quasi-projective): pass to an affine chart, take T_pV as above, take closure in $\mathbb{P}^n$.

Smooth Points

A point p on an affine alg. (or quasi/proj.) var. V is smooth if $\dim T_pV = \dim_p V$.
Otherwise, p is singular.

Say V is smooth if it is smooth at all points.
cf. 27 lines theorem.

Note. This makes sense over any field!

Examples. ① $\mathbb{A}^n$ is smooth at all points since $T_p\mathbb{A}^n = \mathbb{A}^n$.
② As above $Z(x^m - y^m)$ is smooth at all points.

A variety has a smooth locus and a singular locus.

Example. The singular locus of $Z(y^2 - x^3)$ is $(0,0)$.

More generally, the sing. locus is small...

Thm. The singular locus of a variety V is a proper closed subset. More specifically, if V is an irred aff. var. of dim d with $I(V)$ gen. by $f_1, \dots, f_r$, then the sing. locus is the common zero set of the $(n-d) \times (n-d)$ minors of the Jacobian matrix $\left({}^{df_i}/_{dx_j} \right)$.

Pf for $n=2, d=1$. In this case the tangent space is given by
$$\frac{df}{dx}(a,b)(x-a) + \frac{df}{dy}(a,b)(y-b) = 0$$

So sing. locus is $Z(f, {}^{df}/_{dx}, {}^{df}/_{dy})$ $\square$

Next time: a coord. free description:
$$T_p V \cong \left({}^{m}/_{m^2} \right)^*$$

where $m \subseteq k[V]$ is the set of fns that vanish at p.

Cotangent Spaces

A linear form on T_pV is an element of T_p^*V, the dual of T_pV. In other words, a linear form is a linear map $T_pV \longrightarrow k$.

Prop. $V = Z(f_1, \ldots, f_r) \subseteq \mathbb{A}^n$
$p \in V$, $g \in k[V]$.
Then dg is a linear form on T_pV.

Pf. dg is linear. The point is to show it is well defined on T_pV.
Say $G_1, G_2 \in k[x_1, \ldots, x_n]$ map to $g \in k[V]$.
$\Longrightarrow G_1 - G_2 = \sum F_i \cdot f_i$.
$\Longrightarrow d_p(G_1 - G_2) = \sum d_pF_i \, f_i + F_i d_p f_i$
Since $f_i = 0$ on V, first set of terms vanish.
But T_pV is defined by $d_pf_i = 0$.
$\Longrightarrow d_pG_1 = d_pG_2$ $\square$

Let $m \subseteq k[V]$ be the unique max. ideal of functions that vanish at p. If $p = (a_1, \ldots, a_n)$ then

$$m = (x_1 - a_1, \ldots, x_n - a_n).$$

Prop. $V = Z(f_1, \ldots, f_m) \subseteq \mathbb{A}^n$

Differentiation induces a surjective map

$$\mathfrak{m} \longrightarrow T_p^* V$$

with kernel $\mathfrak{m}^2$.

Pf. WLOG $p = 0$.

Let $e_1, \ldots, e_r$ be a basis for T_pV,

extend to basis $e_1, \ldots, e_n$ for $\mathbb{A}^n$.

Assume the f_i are written wrt this basis.

Let e^i be dual basis for $(\mathbb{A}^n)^*$.

Let $M = (x_1, \ldots, x_n)$ max ideal.

Then $\mathfrak{m}$ is image of M in $k[V]$.

Surjectivity. Let $\ell = \sum^r c_i e^i \in T_p^* V$.

This ℓ extends to linear functional on $\mathbb{A}^n$

Let $L = \sum^r c_i x_i \in M$.

The image of L is $k[V]$ has differential ℓ.

Kernel. Say $g \in \mathfrak{m}$ has $dg = 0$. Say g is image of $G \in M$. Then $d_pG \equiv 0$ on T_pV.

Then $d_pG = \sum^m \lambda_j d_pf_j$

Let $\bar{G} = G - \sum \lambda_i f_j$.

Then $\bar{G}$ still maps to g, but $d_p\bar{G} = 0$ on T_pV.

$\Longrightarrow$ const & lin. terms vanish $\Longrightarrow \bar{G} \in M^2$

$\Longrightarrow g \in \mathfrak{m}^2$. $\square$

If R is a ring with max. ideal m, then $R \cdot m \subseteq m$ and $R \cdot m^2 \subset m^2$, so:

m and m/m^2 are modules over R.

Also, multiplication by elts of m gives 0 so:

m/m^2 is a module over the field R/m.

(that is, a vector space).

By the previous prop, we now have:

Thm. $V \subseteq \mathbb{A}^n$ affine alg. var.

$p \in V$

$m \subseteq k[V]$ as above.

Then $T_pV \cong ({}^{m}/_{m^2})^*$

The vector space $({}^{m}/_{m^2})^*$ is sometimes called the *Zariski tangent space*.

Cor. $f: V \to W$ is a morphism of affine alg. var.'s, $p \in V$

Then f induces a lin. map $T_pV \to T_{f(p)}W$

Pf. f induces $g: k[W] \to k[V]$, $g^{-1}(m) = n$ max id. for $f(p)$.

$\rightsquigarrow n/n^2 \to m/m^2$ $\square$

We also get a coordinate free definition of the differential.

Prop. V = irred. affine var, $p \in V$.
$f \in k[V]$
Then $f - f(p) \in m$ and
$$d_p f = \text{image of } f - f(p) \text{ in } m/m^2.$$

Pf. Lift f to $F \in k[x_1, \dots, x_n]$
Subtracting $F(p)$ kills const. term.
Modding out by M^2 kills quadratic and higher.

Example. $V = Z(x^3 - y^2) \subseteq \mathbb{A}^2$.
First we find two poly's rep'ing same elt of $T^*_{(1,1)}V$:
Here, $m = (x-1, y-1)$
$\rightsquigarrow m^2 = (x^2 - 2x + 1,\ (x-1)(y-1),\ y^2 - 2y + 1)$
$= (x^2 - 2x + 1,\ (x-1)(y-1),\ x^3 - 2y + 1$
$\Rightarrow y = (x^3 + 1)/2$
$= (x \cdot (2x-1) + 1)/2$
$= (2x^2 - x + 1)/2$
$= (3x - 1)/2$ in m/m^2.

Next we show V is not smooth at $(0,0)$:
$m = (x, y) \Rightarrow m^2 = (x^2, xy, y^2) = (x^2, xy)$
$\Rightarrow m/m^2$ is vect. sp. spanned by x & y.
But $\dim V = 1$. $\square$