# VARIETIES

Roughly, a variety is a space that is locally isomorphic to an affine variety. Think: manifold.

Def. • A prevariety is a ringed space $X$ that has a finite open cover by affine varieties

• A morphism of prevarieties is a morphism of ringed spaces.

• The elts of $\mathcal{O}_X(U)$ are called regular fns

An open subset of $X$ isomorphic to an aff. alg. var is called an affine open set.

Examples. ① affine alg. var's

② open subsets of aff alg. var's

recall: any open subset of $Z(I)$ is covered by finitely many $D(f)$

Next: can glue pre-varieties together.

# Gluing Pre-varieties

$X_1, X_2$ = pre-varieties
$U_{1,2} \subseteq X_1 \quad U_{2,1} \subseteq X_2$ nonempty open subsets
$f : U_{1,2} \longrightarrow U_{2,1}$ an isomorphism.

$$\rightsquigarrow \quad X = X_1 \coprod_f X_2 \qquad \text{gluing}$$
$$= X_1 \amalg X_2 / (f(a) \sim a)$$

Let $i_j : X_j \longrightarrow X$ be $x \longmapsto [x]$ equiv. class
Say $U \subseteq X$ open if $i_j^{-1}(U)$ open $j=1,2$ (quotient top.)
Define for all open $U \subseteq X$

$$\mathcal{O}_X(U) = \{\varphi : U \to k : i_j^* \varphi \in \mathcal{O}_{X_j}(i_j^{-1}(U)) \ j=1,2\}$$

So: a fn is regular if both restrictions are
This does define a sheaf.

Exercise. Images of $i_1, i_2$ are open subsets of $X$ isomorphic to $X_1, X_2$.
We generally identify $X_1$ & $X_2$ with their images.

another exercise

Since $X_1, X_2$ covered by affine open sets, this is true for $X$. Thus: $X$ is a prevariety.

**Example.** $X_1 = X_2 = \mathbb{A}^1$

$U_{1,2} = U_{2,1} = \mathbb{A}^1 \setminus \{0\}$

We'll consider two different $f$'s.

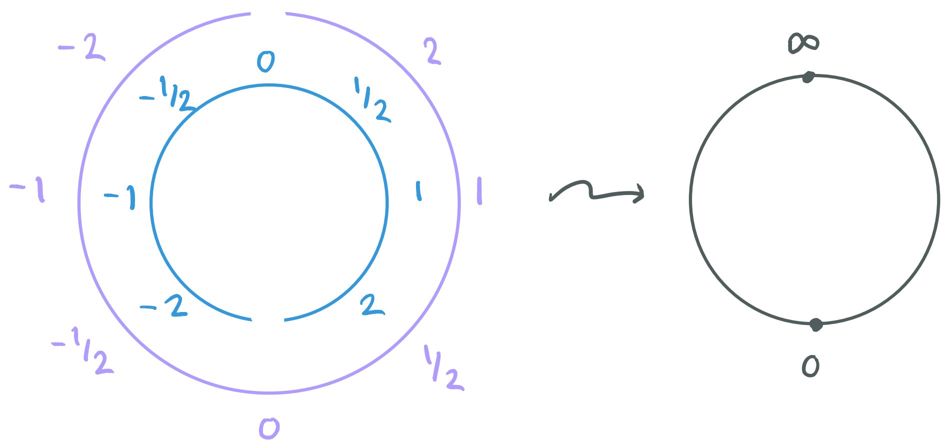
$\boxed{f(x) = 1/x}$ By construction $X_1 = \mathbb{A}^1$ open in $X$.

The complement $X \setminus X_1 = X_2 \setminus U_{2,1}$ is $\{0\} \in X_2$

This corresponds to $\infty = \frac{1}{0}$ in $X_1$

$\rightsquigarrow$ $X = \mathbb{A}^1 \cup \{\infty\}$ $(= \mathbb{P}^1)$.

For $k = \mathbb{C}$ this is $\hat{\mathbb{C}}$. The $\mathbb{R}$-points form a circle:



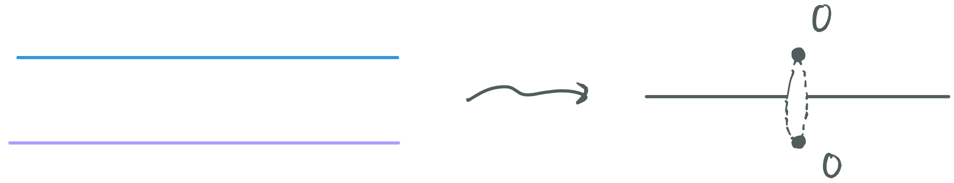

We can give an example of gluing morphisms

$$X_1 \longrightarrow X_2 \subseteq \mathbb{P}^1 \qquad X_2 \longrightarrow X_1 \subseteq \mathbb{P}^1$$
$$x \longmapsto x \qquad\qquad x \longmapsto x$$

These glue together to give the morphism

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^1$$
$$x \longmapsto 1/x$$

$\boxed{f(x) = x}$ In this case get $\mathbb{A}^1$ with two 0's.




The piecewise defined map gives a map $g: X \to X$ that exchanges the two 0's. It is weird that $\mathbb{A}^1 \setminus \{0\}$ is not closed (not even in the Euclidean topology), but it is the set of solutions to $g(x) = x$.

When we finally define a variety, we will rid this pathology.

General gluing construction $I$ = finite set, $X_i$ = pre-var $i \in I$.
Suppose $\forall\, i \neq j$ we have open $U_{ij}$ & isomorphisms $f_{ij}: U_{ij} \longrightarrow U_{ji}$ s.t. $\forall$ distinct $i, j, k$ we have

- $f_{ji} = f_{ij}^{-1}$
- $U_{ij} \cap f_{ij}^{-1}(U_{jk}) \subseteq U_{ik}$ and
  $f_{jk} \circ f_{ij} = f_{ik}$ on $U_{ij} \cap f_{ij}^{-1}(U_{jk})$

$$\rightsquigarrow \quad X = \coprod X_i \big/ a \sim f_{ij}(a)$$

The above conditions ensure $\sim$ is symm & trans.
Now define topology & structure sheaf as before.

**Example** Complex affine curves.

$$X = \{(x,y) \in \mathbb{A}^2_{\mathbb{C}} : y^2 = (x-1)(x-2)\cdots(x-2n)\}$$

Recall this looks like  $(n=3)$

We'd like to compactify, by adding a point $x = \infty$ and two corresponding $y$-values.

Make coord change $\bar{x} = 1/x$ where $x \neq 0$.

$\rightsquigarrow$ $y^2 \bar{x}^{2n} = (1-\bar{x})(1-2\bar{x})\cdots(1-2n\bar{x})$

Also $\bar{y} = yx^n$

$\rightsquigarrow$ $\bar{y}^2 = (1-\bar{x})(1-2\bar{x})\cdots(1-2n\bar{x})$

We can now add the pts $\bar{x} = 0$, $\bar{y} = \pm 1$.

Get a compactified curve by gluing $X_1 = X$ (as above) to $X_2 = \{(\bar{x},\bar{y}) \in \mathbb{A}^2 : \bar{y} = (1-\bar{x})(1-2\bar{x})\cdots(1-2n\bar{x})\}$

with
$$f : U_{1,2} \longrightarrow U_{2,1}$$
$$(x,y) \longmapsto (\bar{x},\bar{y}) = \left(\frac{1}{x}, \frac{y}{x^n}\right)$$

where $U_{1,2} = \{(x,y) : x \neq 0\}$, $U_{2,1} = \{(x,y) : \bar{x} \neq 0\}$

**Next:** Which other operations on pre-varieties (besides gluing) give more pre-varieties?

# Open & Closed Sub-Prevarieties

$X$ = pre-variety.

**Open subprevarieties.** $U \subseteq X$ open. Then $U$ is a pre-var with $\mathcal{O}_U = \mathcal{O}_X|_U$.

Since $X$ is covered by affine varieties, $U$ is covered by open subsets of affine varieties. We already showed these are, in turn, covered by finitely many $D(f)$'s, which are affine varieties.

**Closed subprevarieties.** Let $Y \subseteq X$ closed. An open $U \subseteq Y$ is not nec. open in $X$, so can't define the structure sheaf $\mathcal{O}_Y$ that way. Instead, define $\mathcal{O}_Y(U)$ to be the $k$-alg. of fns $U \longrightarrow k$ that are locally restrictions of sections on $X$:

$$\mathcal{O}_Y(U) = \{\varphi : U \to k : \forall\, a \in U \ \exists \text{ open nbd } V \text{ of } a \text{ in } X \text{ and } \varphi' \in \mathcal{O}_X(V) \text{ s.t. } \varphi = \varphi'|_U\}$$

Exercise: this makes $Y$ a pre-variety.

**Locally closed subprevarieties.** $U$ open, $Y$ closed $\Longrightarrow$ $U \cap Y$ open in $Y$ & closed in $U$. Combine the previous two constructions (there are 2 ways, but get same answer).

Example. $\{(x,y) \in \mathbb{A}^2 : x = 0,\ y \neq 0\} \subseteq \mathbb{A}^2$

For more complicated subsets, we may not be able to make it into a pre-var.

Non-example. $\mathbb{A}^2 - (\{x\text{-axis}\} \setminus 0)$

This does not look like an aff. var. near $0$.

# Products of Pre-varieties

Naively, would cover $X$ & $Y$ by finitely many aff. var's and take the products of those. But would need to check the resulting sheaf is well def.

**Def.** $X, Y$ pre-varieties
A product of $X$ & $Y$ is a prevariety $P$ with morphisms $\pi_X : P \to X$ & $\pi_Y : P \to Y$ s.t.

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] & X \\
& Y &
\end{tikzcd}$$

**Prop.** Any two pre-varieties have a product $P$.
Moreover $P$ with $\pi_X, \pi_Y$ is unique up to $\cong$.

We denote $P$ by $X \times Y$.