VARIETIES

Roughly, a variety is a space that is locally isomorphic to an affine variety. Think: manifold.

Def. • A prevariety is a ringed space $X$ that has a finite open cover by affine varieties
• A morphism of prevarieties is a morphism of ringed spaces.
• The elts of $O_X(U)$ are called regular fns

An open subset of $X$ isomorphic to an aff. alg. var is called an affine open set.

Examples. ① affine alg. var's
② open subsets of aff alg. var's
recall: any open subset of $\mathbb{Z}(I)$ is covered by finitely many $D(f)$

Next: can glue pre-varieties together:
Gluing Pre-varieties

Let \( X_1, X_2 \) be pre-varieties.

\[ U_{1,2} \subseteq X_1, \quad U_{2,1} \subseteq X_2 \]

are nonempty open subsets.

\( f : U_{1,2} \to U_{2,1} \) is an isomorphism.

\( \xrightarrow{\sim} X = X_1 \sqcup \frac{X_2}{(f|_{(a) \sim a})} \)

Let \( i_j : X_j \to X \) for \( j = 1, 2 \).

Say \( U \subseteq X \) is open if \( i_j^{-1}(U) \) is open for \( j = 1, 2 \) (quotient topology).

Define for all open \( U \subseteq X \):

\[ \mathcal{O}_X(U) = \{ \varphi : U \to k : \forall j, i_j^* \varphi \in \mathcal{O}_{X_j}(i_j^{-1}(U)) \} \]

So a \( \varphi \) is regular if both restrictions are.

This does define a sheaf.

Exercise. Images of \( i_1, i_2 \) are open subsets of \( X \) isomorphic to \( X_1, X_2 \).

We generally identify \( X_1 \) & \( X_2 \) with their images.

Since \( X_1, X_2 \) are covered by affine open sets, this is true for \( X \). Thus: \( X \) is a prevariety.
Example. \( X_1 = X_2 = \mathbb{A}' \)

\[ U_{1,2} \cap U_{2,1} = \mathbb{A}' \setminus \{0\} \]

We'll consider two different \( f' \)'s.

\[ f(x) = \frac{1}{x} \]

By construction \( X_1 = \mathbb{A}' \) open in \( X \).

The complement \( X \setminus X_1 = X_2 \setminus U_{2,1} \) is \( \{0\} \in X_2 \)

This corresponds to \( \infty = \frac{1}{0} \) in \( X_1 \)

\[ \sim X = \mathbb{A}' \cup \{\infty\} (= \mathbb{P}') \]

For \( k = \mathbb{C} \) this is \( \hat{\mathbb{C}} \). The \( \mathbb{R} \)-points form a circle:

We can give an example of gluing morphisms

\[ X_1 \to X_2 \subseteq \mathbb{P}' \quad X_2 \to X_1 \subseteq \mathbb{P}' \]

\[ x \mapsto x \quad x \mapsto x \]

These glue together to give the morphism

\[ \mathbb{P}' \to \mathbb{P}' \]

\[ x \mapsto \frac{1}{x} \]
In this case get $\mathbb{A}'$ with two $O$'s.

\[ f(x) = x \]

The piecewise defined map gives a map $g : X \to X$ that exchanges the two $O$'s. It is weird that $\mathbb{A}' \setminus \{0\}$ is not closed (not even in the Euclidean topology), but it is the set of solutions to $g(x) = x$.

When we finally define a variety, we will rid this pathology.

**General gluing construction**  
$I = \text{finite set, } X_i = \text{pre-var } i \in I.$

Suppose $i \neq j$, we have open $U_{ij}$ & isomorphisms $f_{ij} : U_{ij} \to U_{ji}$ s.t. $\forall$ distinct $i, j, k$ we have

- $f_{ji} = f_{ij}^{-1}$
- $U_{ij} \cap f_{ij}^{-1}(U_{jk}) \subseteq U_{ik}$ and $f_{jk} \circ f_{ij} = f_{ik}$ on $U_{ij} \cap f_{ij}^{-1}(U_{jk})$

$\sim X = \coprod X_i / a \sim f_{ij}(a)$

The above conditions ensure $\sim$ is symm & trans.
Now define topology & structure sheaf as before.
**Example** Complex affine curves.

\[ X = \{(x,y) \in \mathbb{A}^2_c : y^2 = (x-1)(x-2) \cdots (x-2n)\} \]

Recall this looks like \( \cdots \) \( (n=3) \)

We'd like to compactify, by adding a point \( x = \infty \) and two corresponding \( y \)-values.

Make coord change \( \bar{x} = \frac{1}{x} \) where \( x \neq 0 \).

\[ y^2 \bar{x}^{2n} = (1-\bar{x})(1-2\bar{x}) \cdots (1-2n\bar{x}) \]

Also \( \bar{y} = yx^n \)

\[ \bar{y}^2 = (1-\bar{x})(1-2\bar{x}) \cdots (1-2n\bar{x}) \]

We can now add the pts \( \bar{x} = 0, \bar{y} = \pm 1 \).

Get a compactified curve by gluing \( X_1 = X \) (as above) to \( X_2 = \{(\bar{x},\bar{y}) \in \mathbb{A}^2 : \bar{y} = (1-\bar{x})(1-2\bar{x}) \cdots (1-2n\bar{x})\} \)

with \( f : U_{1,2} \rightarrow U_{2,1} \)

\[ (x,y) \mapsto (\bar{x},\bar{y}) = (\frac{1}{x}, \frac{y}{x^n}) \]

where \( U_{1,2} = \{(x,y) : x \neq 0\} \) , \( U_{2,1} = \{(x,y) : \bar{x} \neq 0\} \)

**Next:** Which other operations on pre-varieties (besides gluing) give more pre-varieties?
Open & Closed Sub-prevarieties  \( X = \text{pre-variety} \)

**Open subprevarieties.**  \( U \subseteq X \) open. Then \( U \) is a pre-var with \( \mathcal{O}_U = \mathcal{O}_X|U \).

Since \( X \) is covered by affine varieties, \( U \) is covered by open subsets of affine varieties. We already showed these are, in turn, covered by finitely many \( \text{D}(f) \)'s, which are affine varieties.

**Closed subprevarieties.** Let \( Y \subseteq X \) closed. An open \( U \subseteq Y \) is not nec. open in \( X \), so can't define the structure sheaf \( \mathcal{O}_Y \) that way. Instead, define \( \mathcal{O}_Y(U) \) to be the \( k \)-alg. of \( \text{fns} \ U \rightarrow k \) that are locally restrictions of sections on \( X \):

\[
\mathcal{O}_Y(U) = \left\{ \phi : U \rightarrow k : \forall a \in U \exists \text{ open nbd } V \text{ of } a \text{ in } X \text{ and } \phi' \in \mathcal{O}_X(V) \text{ s.t. } \phi = \phi'|U \right\}
\]

Exercise: this makes \( Y \) a pre-variety.

**Locally closed subprevarieties**  \( U \) open, \( Y \) closed \( \Rightarrow U \cap Y \) open in \( Y \) & closed in \( U \). Combine the previous two constructions (there are 2 ways, but get same answer).

**Example.** \( \{(x,y) \in \mathbb{A}^2 : x=0, y \neq 0\} \subseteq \mathbb{A}^2 \)
For more complicated subsets, we may not be able to make it into a pre-var.

Non-example. \( \mathbb{A}^2 - (\{x\text{-axis}\} \setminus \{0\}) \)
This does not look like an aff. var. near 0.
**Products of Pre-varieties**

Naively, would cover $X$ & $Y$ by finitely many aff. var's and take the products of those. But would need to check the resulting sheaf is well def.

**Def.** $X, Y$ pre-varieties

A **product** of $X$ & $Y$ is a prevariety $P$ with morphisms $\pi_X : P \to X$ & $\pi_Y : P \to Y$ s.t.

![Diagram](image)

**Prop.** Any two pre-varieties have a product $P$. Moreover $P$ with $\pi_X, \pi_Y$ is unique up to $\simeq$.

We denote $P$ by $X \times Y$. 