MATH 137 NOTES: UNDERGRADUATE ALGEBRAIC GEOMETRY

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CONTENTS

1. Introduction

Joe Harris taught a course (Math 137) on undergraduate algebraic geometry at Harvard in Spring 2016.

These are my “live-TEXed“ notes from the course. Conventions are as follows: Each lecture gets its own “chapter,” and appears in the table of contents with the date.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are my fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe. [1] Several of the classes have notes taken by Hannah Larson. Thanks to her for taking notes when I missed class.

Please email suggestions to `aaronlandesman@gmail.com`.

2. Conventions

Here are some conventions we will adapt throughout the notes.

(1) I have a preference toward making any detail stated in class, which is not verified, into an exercise. This will mean there are many trivial exercises in the notes. When the exercise seems nontrivial, I will try to give a hint. Feel free to contact me if there are any exercises you do not know how to solve or other details which are unclear.
(2) Throughout the notes, I will often include parenthetical remarks which describe things beyond the scope of the course. If some word or explanation is placed in quotation marks or in parentheses, and you don’t understand it, don’t worry about it! It’s more meant to give you a flavor of how one might describe the 19th century ideas in this course in terms of 20th century algebraic geometry.

[1]This introduction has been adapted from Akhil Matthew’s introduction to his notes, with his permission.

3. 1/25/16

3.1. **Logistics.**

(1) Three lectures a week on Monday, Wednesday, and Friday
(2) We'll have weekly recitations
(3) This week we'll have special recitations on Wednesday and Friday at 3pm;
(4) You're welcome to go to either section
(5) Aaron will be taking notes
(6) Please give feedback to Professor Harris or the CAs
(7) We'll have weekly problem sets, due Fridays. The first problem set might be due next Wednesday or so.
(8) The minimal prerequisite is Math 122.
(9) The main text is Joe Harris' "A first course in algebraic geometry."
(10) The book goes fast in many respects. In particular, the definition of projective space is given little attention.
(11) Some parts of other math will be used. For example, knowing about topology or complex analysis will be useful to know, but we'll define every term we use.

3.2. **History of Algebraic Geometry.** Today, we will go through the History of algebraic geometry.

3.3. **Ancient History.** The origins of algebraic geometry are in understanding solutions of polynomial equations. People first studied this around the late middle ages and early renaissance. People started by looking at a polynomial in one variables $f(x)$. You might then want to know:

Question 3.1. What are the solutions to $f(x) = 0$?

(1) If $f(x)$ is a quadratic polynomial, you can use the quadratic formula.
(2) For some time, people didn't know how to solve cubic polynomials. But, after some time, people found an explicit formula for cubics.
(3) In the beginning of the 19th century, using Galois theory, people found this pattern does not continue past degree 5, and there are no closed form solutions in degree 5 or more.

So, people jumped to the next level of complexity: polynomials in two variables.

Question 3.2. What are the solutions to $f(x, y) = 0$? Or, what are the simultaneous solutions to a collection of polynomials $f_i(x, y) = 0$ for all i?

Algebraic geometry begins here.

Goal 3.3. The goal of algebraic geometry is to relate the algebra of f to the geometry of its zero locus.

This was the goal until the second decade of the nineteenth century. At this point, two fundamental changes occurred in the study of the subject.

3.3.1. *Nineteenth century.* In 1810, Poncelet made two breakthroughs. Around this time, Poncelet was captured by the Russians, and held as a prisoner. In the course of his captivity, he found two fundamental changes.

(1) Work over the complex numbers instead of the reals.
(2) Work in projective space (to be described soon), instead of affine space, that is, in $\mathbb{C}^n$.

There are several reasons for these changes. Let's go back to the one variable case. The reason the complex numbers are nice is the following:

Theorem 3.4 (Fundamental Theorem of Algebra)**.** *A polynomial of degree n always has n solutions over the complex numbers.*

Example 3.5. This is false over the real numbers. For example, consider $f(x) = x^2 + 1$.

This makes things nicer over the complex numbers. If we have a family of polynomial equations (meaning that we vary the coefficients) then the number of solutions will be constant over the complex numbers.

Let's see two examples of what can go wrong, and how Poncelet's two changes help this. The motivation is to preserve the number of intersection of some algebraic varieties.

(1)

Example 3.6. Suppose we have a line and a conic (e.g., a circle, or something cut out by a degree 2 polynomial in the projective plane). If the line meets conic, over the real numbers, there will be 0, 1, or 2 solutions, depending on where the line lies relative to the circle.

(2)

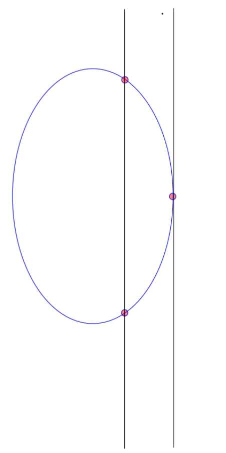


FIGURE 1. As the lines limit away from the ellipse over the real numbers, they start with two intersection points, limit to 1, and then pass to not intersect at all. This situation is rectified when we work over the complex numbers (although we will have to "count with multiplicity").

Example 3.7. If we have two lines which are parallel, they won't meet. But, if they are not parallel will meet. So, we need to add a "point at infinity." This motivates using projective space, which does include these points.

Remark 3.8. By working over the complex numbers and in projective space, you lose the ability to visualize some of these phenomena. However, you gain a lot. For example, heuristically speaking, you obtain that the "number of solutions" will be constant in nice (flat) families.

Let's see an example:

Example 3.9. Consider $x^2 + y^2$ and $x^2 - y^2$, which look very different over the reals. The first only has a zero at the origin, while the second is two lines. However, over the complex numbers, they are related by a change of variables $y \mapsto iy$.

In fact, in projective space, the hyperbola, parabola, and ellipse are essentially the same. The only difference is how they meet the line at infinity over the real numbers. (Over the complex numbers, they

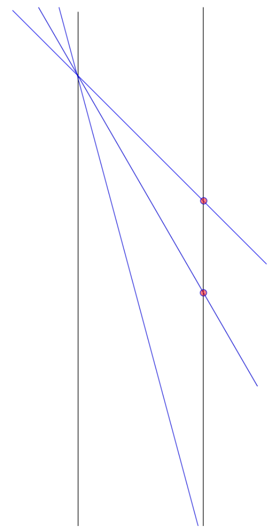


FIGURE 2. As the blue lines limit to the left black line, their point of intersection limits down the right black line to the point at infinity. Hence, we need to include this point at infinity so that two parallel lines will meet.

will all meet the line at infinity "with multiplicity two.") Over the reals, a parabola is tangent to the line at infinity, a hyperbola meets the line at infinity twice, and an ellipse doesn't meet it.

3.4. **Twentieth Century.** In 1900 to 1950, there was an algebraicization of algebraic geometry, by Zariski and Weil.

Between 1950 and the present, there was the development of what we think of as modern algebraic geometry. There was in introduction of schemes, sheaves, stacks, etc. This was due, among others, to Grothendieck, Serre, Mumford, Artin, etc.

Remark 3.10. Once again, this led to a trade off. This led to a theory with much greater power. However, the downside is that you lose a lot of intuition which you used to have over the complex numbers. If one wants to do algebraic geometry, one does need to learn the modern version. However, jumping in and trying to learn this, is like beating your head against the wall.

3.5. **How the course will proceed.** We will stick to the 19th century version of algebraic geometry in this course. We won't shy away from using things like rings and fields, but we will not use any deep theorems from commutative algebra.

3.6. Beginning of the mathematical portion of the course.

Remark 3.11. For the remainder of this course, we'll work with a field $\mathbb{k}$ which is algebraically closed and of characteristic 0. If you prefer, you can pretend $\mathbb{k} = \mathbb{C}$.

Definition 3.12. We define **affine space** of dimension n, notated $\mathbb{A}^n_{\mathbb{k}}$, or just $\mathbb{A}^n$ when the field of definition $\mathbb{k}$ is understood, is the set of points

$$\mathbb{k}^n = \{(x_1, \ldots, x_n), x_i \in k\}.$$

Remark 3.13. The subtle difference between $\mathbb{k}^n$ and $\mathbb{A}^n$ is that $\mathbb{A}^n$ has no distinguished 0 point.

Definition 3.14. An affine variety $X \subset \mathbb{A}^n$ is a subset of $\mathbb{A}^n$ describable as the common zero locus of a collection of polynomials

$$\{f_\alpha(x_1, \ldots, x_n)\}_{\alpha \in A}$$

If $f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)$ are polynomials, then an algebraic is

$$V(f_1, \ldots, f_k) = \{x \in \mathbb{A}^n : f_\alpha(x) = 0, \text{ for all } \alpha\}.$$

Remark 3.15. We needn't assume that there are only finitely many polynomials here. The reason is that this variety $V(f_i)$ only depends on the ideal generated by the f_i. Since every ideal over a field is finitely generated (using commutative algebra, which says that algebras of finite type over a field are Noetherian) we can, without loss of generality, restrict to the case that we have finitely many polynomials.

Next, we move onto a discussion of projective space.

Definition 3.16. We define **projective space**

$$\mathbb{P}^n_{\mathbb{k}} = \left\{ \text{one dimensional linear subspaces of } \mathbb{k}^{n+1} \right\}.$$

We often notation $\mathbb{P}^n_{\mathbb{k}}$ as just $\mathbb{P}^n$ when the field of definition $\mathbb{k}$ is understood.

Remark 3.17. An alternative, equivalent definition of projective space is as

$$\mathbb{k}^{n+1} \setminus 0 / \mathbb{k}^\times.$$

where the quotient is by the action of $\mathbb{k}^\times$ by scalar multiplication.

Definition 3.18. We use $(x_1, \ldots, x_n)$ to denote a point in affine space, and use $[x_0, x_1, \ldots, x_n]$ to denote a point, (which is only defined up to scaling) in projective space.

Lemma 3.19. *Consider the subset*

$$U = \{[z] : z_0 \neq 0\}$$

The, $U \cong \mathbb{A}^n$.

Proof. This is well defined because if the first entry is 0, it will still be 0 after scalar multiplication. The map between them is given by

$$\begin{aligned} U &\to \mathbb{A}^n \\ [z_0, \ldots, z_n] &\mapsto \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right) \end{aligned}$$

with inverse map given by

$$\begin{aligned} \mathbb{A}^n &\to U \\ (z_1, \ldots, z_n) &\mapsto [1, z_1, \ldots, z_n] \end{aligned}$$

Exercise 3.20. Check these two maps are mutually inverse, and hence define an isomorphism.

□

Remark 3.21. To make projective space, we start with affine space, $\mathbb{A}^2$, and add in a point for each "direction." For every maximal family of parallel lines, we throw in one additional point on the line at infinity. Now, parallel lines meet at this point, accomplishing the original goal of Poncelet.

Remark 3.22. Notice that the role of the line at ∞ in this case is dependant on the choice of coordinate z_0. So, projective space is actually covered by affine spaces. It is covered by $n+1$ subsets, where we replace z_0 by z_i for $0 \leq i \leq n$. The resulting "standard affine charts" U_i cover $\mathbb{P}^n$.

Further, we could have even chosen any nonzero homogeneous linear function. The set where that function is nonzero is again isomorphic to affine space. So, in this case, the role of the line at infinity could be played by any line at all.

So, when we work in projective space, we don't see any difference between hyperbolas, parabolas, and ellipses.

4. 1/27/16

4.1. **Logistics and Review.** Logistics:

- Hannah is giving section Wednesday at 3 in science center 304
- Aaron is giving a section on Friday meeting in the math common room at 4:30

Fun facts from commutative algebra

(1) (Hilbert Basis Theorem, 1890) Every ideal in $\mathbb{k}[x_1, \ldots, x_n]$ is finitely generated.
(2) The ring $\mathbb{k}[x_1, \ldots, x_n]$ is a UFD.

Remark 4.1. We will not prove the above two remarks, but it would take a day or two in class to prove it. Instead, we'll assume them so we can get to work with varieties right away. This is following the 19th century mathematicians who clearly knew this fact, who similarly used it implicitly without proving it. You can find these facts in Atiyah McDonald or Dennis Gaitsgory's Math 123 notes.

We now recall some things from last time.

(1) First, recall the convention that we are assuming $\mathbb{k}$ is an algebraically closed field of characteristic 0. Joe invites you to pretend $\mathbb{k} = \mathbb{C}$, as in some sense, anything true over one is true over the other.
(2) Any time we say subspace, we mean linear subspace, unless otherwise specified.
(3) We define affine space

$$\mathbb{A}^n := \{(x_1, \ldots, x_n) : x_i \in \mathbb{k}\}$$

and define an affine variety as one of the form

$$Z = V(f_1, \ldots, f_k) = \{x \in \mathbb{A}^n : f_\alpha(x) = 0 \text{ for all } \alpha\}.$$

(4)

We next come to a definition of projective space.

Lemma 4.2. *The following three descriptions are equivalent.*

(1) We define projective space

$$\begin{aligned}\mathbb{P}^n &= \{[x_0, \ldots, x_n] : x \neq 0\} / \mathbb{k}^\times \\ &= \left\{ \text{one dimensional linear subspaces of } \mathbb{k}^{n+1} \right\}\end{aligned}$$

(2) We now make an alternative definition of projective space which is independent of basis . Given a vector space V *which is* $(n+1)$

dimensional over $\mathbb{k}$, *we define*

$$\mathbb{P}V = \{ \text{ on dimensional subspaces of } V\} = (V \setminus \{0\}) / \mathbb{k}^\times$$

(3) *We define*

$$U_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\}$$

which is isomorphic to $\mathbb{A}^n$, *as we saw last time. More generally, if* $L : V \to \mathbb{k}$ *is any homogeneous nonzero linear polynomial, we define*

$$U_L = \{x \in \mathbb{P}^n : L(X) \neq 0\} \cong \mathbb{A}^n$$

Then, we can glue the U_L *together to form projective space.*

Proof. Omitted. □

Remark 4.3. Technically speaking, the last description relies on the first two. However, we can indeed give a definition of projective space as copies of affine space glued together, although this takes a little more work (but not too much more).

Under this description, projective spaces is a sort of compactification of affine space.

The last description, in more advanced terms, is known as the Proj construction, but don't worry about this for this class.

Definition 4.4. We define **projective space** as the space defined in any of the three equivalent formulations as given in Lemma 4.2.

Example 4.5. Consider the polynomial $f(x, y) = 0$ with $f(x, y) = x^2 - y^2 - 1$. This maps to $\mathbb{A}^1_x$ by taking the x coordinate, and we can identify $\mathbb{A}^1_x$ with the points $\mathbb{C}$. In fact, this variety can be viewed as a twice punctured sphere. See Gathman's algebraic geometry notes at http://www.mathematik.uni-kl.de/ gathmann/class/alggeom-2002/main.pdf for some pictures.

Remark 4.6 (A description of low dimensional projective spaces)**.** Recall we can identify $\mathbb{A}^1 = \mathbb{C}$ and $\mathbb{P}^1 = \left\{[x_0, x_1]] / \mathbb{C}^\times\right\}$. Then, inside $\mathbb{P}^1$, we have

$$U = \{[x] : x_0 \neq 0\} \cong \mathbb{A}^1$$

identified via the map

$$U \to \mathbb{A}^1$$

$$[x_0, x_1] \mapsto \frac{x_1}{x_0} \in \mathbb{C}$$

So, $\mathbb{P}^1$ is the one point compactification of $\mathbb{P}^1$.

However, if we consider the same construction for $\mathbb{P}^2$ its already a little more complicated. We can take

$$\mathbb{P}^2 \subset U = \left\{[x] \in \mathbb{P}^2 : x_0 \neq 0\right\} \cong \mathbb{A}^2$$

Here, as a set, we have $\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1$. However, this is a little harder to visualize, especially over the complex numbers.

Exercise 4.7 (Non-precise exercise)**.** Try to visualize $\mathbb{P}^2_{\mathbb{C}}$.

Definition 4.8. If we write $\mathbb{P}^n = \left\{[x] \in \mathbb{k}^{n+1} : x \neq 0\right\} / \mathbb{k}^\times$, the x_α are called homogeneous coordinates on $\mathbb{P}^n$. The ratios $\frac{x_i}{x_j}$, defined on $U_j \subset \mathbb{P}^n$, where U_j is the locus where $x_j \neq 0$, are **affine coordinates**.

Warning 4.9. This notation is bad! The x_α are not functions on $\mathbb{P}^n$ because they are only defined up to scalar multiplication. So, x_α are not functions, and polynomials in the x_α are not functions.

However, there is some good news:

(1) The pairwise rations $\frac{x_i}{x_j}$ are well defined functions on the open subset $U_j \subset \mathbb{P}^n$ where $x_j \neq 0$.
(2) Given a polynomial $F(x_0, \ldots, x_n)$, a homogeneous polynomial in the projective coordinates, $F(X) = 0$ is a well defined.

Exercise 4.10. Show this is indeed well defined. *Hint:* If you multiply the variables by a fixed scalar, the value of the function changes by a power of that scalar. You will crucially use homogeneity of F.

Definition 4.11. A **projective variety** is a subset of $\mathbb{P}^n$ describable as the common zero locus of a collection of homogeneous polynomials

$$V(\{F_\alpha\} = \{X \in \mathbb{P}^n : F_\alpha(X) = 0 \text{ for all } \alpha\}.$$

Exercise 4.12. If $Z = V(\{F_\alpha\})$ is a projective variety, and $U = \{x \in \mathbb{P}^n : x_0 \neq 0\}$, show $Z \cap U$ is the affine variety $V(\{f_\alpha\})$, where

$$f_\alpha = F(1, x_1, \ldots, x_n).$$

.

Definition 4.13. Suppose $f(x_1, \ldots, x_n)$ is any polynomial of degree d, so that

$$f(x) = \sum_{i_1 + \cdots + i_n \leq d} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$$

we define the **homogenization** of f as

$$\begin{aligned} F(x) &= \sum a_{i_1,\ldots,i_n} x_0^{d-\sum a_i} x_1^{a_1} \cdots x_n^{a_n} \\ &= x_0^d \cdots f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0} \end{aligned}$$

Remark 4.14. Now, using the concept of homogenization, we have a sort of converse to Exercise 4.12. If we start with a subset $X \subset \mathbb{P}^n$ of projective space so that its intersection with each standard affine chart U_i is an affine variety then then X is a projective variety.

Remark 4.15. We move a bit quickly to projective space. This is because most of the tools we have in algebraic geometry apply primarily to projective varieties. So, often when we start with an affine variety, we will take the closure and put it in projective space to study it as a projective variety.

Example 4.16 (Linear subspaces)**.** Suppose we have a linear subspace $W \subset V$ with $W \cong \mathbb{k}^{k+1}$ and $V \cong \mathbb{k}^{n+1}$. We obtain an inclusion

$$\begin{array}{ccc} \mathbb{P}W & \hookrightarrow & \mathbb{P}^V \\ \downarrow & & \downarrow \\ \mathbb{P}^k & \longrightarrow & \mathbb{P}^n \end{array} \tag{4.1}$$

where the vertical arrows are isomorphisms. The image of $\mathbb{P}W$ in $\mathbb{P}V$ is called a **linear subspace of** $\mathbb{P}V$. When

(1) $k = n-1$, the image is called a **hyperplane**
(2) $k = 1$, the image is called **line**
(3) $k = 0$, the image is called a **point**

In fact, this shows that any two points determine a line, as the two points correspond to the two basis vectors for the two dimensional subspace W.

Next, we look at finite sets of points.

Lemma 4.17. *If $\Gamma \subset \mathbb{P}^n$ is a finite set of points, then Γ is a projective variety.*

Proof. We claim that given any point $q \in \mathbb{P}^n$, so that $q \notin \Gamma$ there exists a homogeneous polynomial F so that $F(p_i) = 0$ for all i and $F(q) \neq 0$. This suffices, because the intersection of all such polynomials as q ranges over all points other than the p_i will have zero locus which is Γ.

To do this, we will take things one point at a time. For all i, we can choose a hyperplane $H_i = V(L_i)$ so that $H_i \ni p$ but $q \notin H_i$. Then, take $F = \prod L_i$, and the resulting polynomial vanishes on all of the p_i but not in q.

□

Remark 4.18. Examining the proof of Lemma 4.17, we see that we can in fact describe Γ as the common zero locus of polynomials of degree d.

Question 4.19. Can we describe Γ as the common zero locus of polynomials of some degree less than d?

In an appropriate sense, this question is in fact still open!
We have the following special case:

Question 4.20. Can we express Γ as the common zero locus of polynomials of degree $d - 1$.

Example 4.21 (Hypersurfaces)**.** If f is any homogeneous polynomial of degree d on $\mathbb{k}^{n+1}$ then $V(F) \subset \mathbb{P}^n$ is a **hypersurface**.

5. 1/29/16

5.1. **Logistics and review.** Logistics:

(1) Problem 11 on homework 1 will be replaced by a simpler version

Review:

(1) We had some examples of projective varieties.
(2) We saw linear spaces $\mathbb{P}^k \subset \mathbb{P}^n$.
(3) Finite subsets (and asked what degree polynomials you need to describe a finite subset)
(4) Hypersurfaces, defined as $V(F) \subset \mathbb{P}^n$.

Part of what we'll be doing is developing a roster of examples because

(1) examples will be the building blocks of what we work on and
(2) these will describe the sorts of questions we'll ask.

5.2. **Twisted Cubics.** The first example beyond the above examples are twisted cubic curves and, more generally, rational normal curves. Let's start by examining twisted cubics.

But first, we'll need some notation.

Definition 5.1. Suppose $V \cong \mathbb{k}^{n+1}$ is a vector space with $\mathbb{P}V \cong \mathbb{P}^n$. Then, the **group of automorphisms** of $\mathbb{P}^n$, denoted $GL_{n+1}(\mathbb{k})$ acts on $\mathbb{P}V$ by the induced action on V. More precisely, for $\psi \in GL_{n+1}(\mathbb{k})$ the resulting action is given by

$$\begin{aligned} \phi : \mathbb{P}^n &\to \mathbb{P}^n \\ [x_0, \ldots, x_n] &\mapsto [\phi(x_0), \ldots, \phi(x_n)] \end{aligned}$$

Remark 5.2. The action is not faithful (meaning some elements act the same way). In particularly, scalar multiples of the identity all act trivially on $\mathbb{P}^n$.

Definition 5.3. Since $\mathbb{k}^* \subset GL_{n+1}$, the invertible scalar multiples of the identity act trivially on $\mathbb{P}^n$, we define the **projective general linear group**

$$PGL_{n+1}(\mathbb{k}) = PGL(V) = GL_{n+1}(\mathbb{k})/\mathbb{k}^\times.$$

For brevity, when the field is understood we notate $PGL_{n+1}(\mathbb{k})$ as PGL_{n+1}. We say two subsets $X, X' \subset \mathbb{P}^n$ are projectively equivalent if there is some $A \in PGL_{n+1}(\mathbb{k})$ with $A(X') = X$.

Definition 5.4. Consider the map

$$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{t \mapsto (t,t^2,t^3)} & \mathbb{A}^3 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{[x_0,x_1] \mapsto [x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3]} & \mathbb{P}^3 \end{array} \tag{5.1}$$

A **twisted cubic** is any variety $C \subset \mathbb{P}^3$ which is projectively equivalent to the image of the bottom map.

Remark 5.5. Even though the x_i are not really functions on projective space, the bottom map is well defined as a map of projective spaces because all the polynomials are homogeneous of the same degree.

Remark 5.6. One can alternatively phrase the definition of a twisted cubic as the image of any map

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ [x] &\mapsto [F_0(x), F_1(x), F_2(x), F_3(x)] \end{aligned}$$

where $F_0, \ldots, F_3$ is a basis for the vector space of homogeneous cubic polynomials on $\mathbb{P}^1$.

Remark 5.7. There is a useful theorem which we will see soon. If we take any map defined in terms of polynomials between two projective spaces, the image will be a projective variety, meaning it will be the common zero locus of polynomial equations.

Lemma 5.8. *A twisted cubic is a variety.*

Proof. It suffices to write down polynomials in four coordinates whose vanishing locus is

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ [x_0, x_1] &\mapsto \left[x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3\right]. \end{aligned}$$

We have that the polynomials

$$\begin{aligned} z_0 z_2 - z_1^2 \\ z_0 z_3 - z_1 z_2 \\ z_1 z_3 - z_2^2 \end{aligned}$$

which certainly contain the image.

Exercise 5.9. Verify that the intersection of these polynomials is precisely the twisted cubic. *Hint:* Essentially, these equation characterize the ratios between the image coordinates as functions of x_0, x_1. For a more advanced and general technique, look up Gröbner bases.

□

Remark 5.10. The quadratic polynomials in the proof of Lemma 5.8 in fact span the space of all quadratic polynomials in $z_0, \ldots, z_3$ vanishing on the twisted cubic.

To see this, we have a pullback map

$$\begin{array}{c} \{\text{ homogeneous quadratic polynomials in } z_0, \ldots, z_3\} \\ \downarrow \\ \{\text{ homogeneous sextic polynomials in } x_0, x_1\} \end{array} \tag{5.2}$$

The way we get this map is by taking a homogeneous polynomial in the z_i and plug in $x_0^i x_1^{3-i}$. This map is surjective because if we have any monomial of degree 6 in x_0 and x_1, we can write it as a pairwise product of two cubics. So, the map goes from a 10 dimensional vector space to a 7 dimensional vector space, and so the kernel is precisely 3 dimensional.

In fact, these three quadrics generate the ideal of all polynomials vanishing on the twisted cubic. This requires more work. For example, it follows from the theory of Gröbner bases.

Warning 5.11. Often, we will conflate vector spaces and generators for their basis. That is, when we say the three quadratic polynomials are all quadrics vanishing on the variety, we really mean the vector space spanned by contains all quadrics vanishing on the variety.

Warning 5.12. Just because some ideal defines a variety, this does not mean it contains all polynomials vanishing on the variety. For example, if $X = V(I^2)$ then we also have $X = V(I)$.

Question 5.13. Do we need all three quadratic polynomials to generate the homogeneous ideal of the twisted cubic?

In fact, we do need all three quadratic polynomials. If we take any linear combination of two of the three quadratic polynomials, we will obtain the union of a twisted cubic and a line. However, this may take a fair amount of computation. (It can also be shown by an advanced tool such as Bezout's theorem, which we do not yet have access to.)

Definition 5.14. A **rational normal curve**$C \subset \mathbb{P}^n$ is defined to be the variety in projective space linearly equivalent to the image of the map

$$\mathbb{P}^1 \to \mathbb{P}^n$$
$$[x_0, x_1] \mapsto \left[x_0^n, x_0^{n-1}x_1, \ldots, x_1^n\right]$$

Equivalently, one can define a rational normal curve as the image of a map

$$[x] \mapsto [f_0(x), \ldots, f_n(x)]$$

where $f_0, \ldots, f_n$ form a basis for the homogeneous polynomials of degree n on $\mathbb{P}^1$.

Example 5.15. The twisted cubic is a rational normal curve for $n = 3$.
When $n = 2$ the map is given by

$$\mathbb{P}^1 \to \mathbb{P}^2$$
$$[x_0, x_1] \mapsto \left[x_0^2, x_0x_1, x_1^2\right]$$

and the image is $V(z_0z_2 - z_1^2)$. This is just a plane conic, and is already covered by the case of hypersurfaces. Conversely, we'll see that every plane conic is a rational normal curve.

Exercise 5.16. The homogeneous ideal of a rational normal curve (when we choose the forms to be $x_0^i x_1^{n-i}$ in $\mathbb{P}^n$ is generated by the quadratic polynomials of the form $x_i x_j - x_k x_l$ where $i + j = k + l$. *: Hint* Follow the analogous proof as for twisted cubics.

5.3. **Basic Definitions.** Whenever we're working on 20th century mathematics, we should really acknowledge the existence of category theory. That is, we should explain the objects we're working with and what the morphisms are. So, we'll soon define the category of quasi-projective varieties. But, before that, we'll need to define open an closed sets. More precisely, we'll be working in a new and beautiful topology called the Zariski topology.

Definition 5.17. Let X be a variety. The **Zariski topology** on X is the topology whose closed subsets are subvarieties of X.

Exercise 5.18. Verify that the Zariski topology is indeed a topology. That is, verify that arbitrary intersections and finite union of closed sets are again closed sets, and verify that the empty set and X are closed.

Warning 5.19. The Zariski topology is not particularly nice. (For example, if you're familiar with the notation, the Zariski topology is T_0 but not T_1). For example, if $X = \mathbb{A}^1$, the Zariski topology is the cofinite topology. That is, the closed sets are only the whole $\mathbb{A}^1$ and finite sets.

In fact, for $\mathbb{A}^1$, any bijection of sets $\mathbb{A}^1 \to \mathbb{A}^1$ is continuous. But, any two open sets intersect.

Remark 5.20. Note that if you're dealing with a variety over an arbitrary field, you don't a priori have a topology on that field. Nevertheless, this is a very useful notion for working over arbitrary fields.

We'll now move on from chapter 1 to chapter 2. Feel free to read through chapter 1, but don't try to parse every sentence. Just concentrate on what we discussed in class because there is too much stuff in our textbook.

5.4. **Regular functions.** We'll first discuss this for affine varieties $X \subset \mathbb{A}^n$ and then for projective varieties.

Definition 5.21. Let $X \subset \mathbb{A}^n$ be an affine variety, and $U \subset X$ is an open subset. Then, a **regular function** on U is a function $f : U \to \Bbbk$ so that for all points $p \in U$, there exists a neighborhood $p \in V$ so that on V, if there exist polynomials $g(x), h(x)$ so that we can write

$$f(x) = \frac{g(x)}{h(x)}$$

for all $x \in V$, where $h(p) \neq 0$.

Remark 5.22. You may have to change the polynomials g and h, depending on what the point p.

6. 2/1/16

6.1. **Logistics and Review.** Logistics

(1) For sectioning, take the poll on the course web page.
(2) This week, the sections will be the same time and place as last week.
(3) On the homework, we have taken out the last problem. It is due this Wednesday
(4) The next homework will be due next Friday, and there will be weekly homework each Friday following.

6.2. **The Category of Affine Varieties.** We return to algebraic geometry, introducing the objects and morphisms of our algebraic category. Recall from last week, we defined the Zariski topology.

Definition 6.1. A **quasi-projective variety** is a open subset of a projective variety (open in the Zariski topology).

Remark 6.2. The class of quasi-projective varieties includes all projective and affine varieties. If we have any affine variety, we can view it as the complement of a hyperplane section of a projective variety via the inclusion $\mathbb{A}^n \to \mathbb{P}^n$.

Heuristically, you can think of this as taking some polynomials equal to 0 and other polynomials which you set to not be equal to 0.

Warning 6.3. Not all quasi-projective varieties are affine or projective.

Exercise 6.4. Show that $\mathbb{A}^1 \setminus \{0\}$ is neither projective nor affine as a subset of $\mathbb{P}^1$.

Exercise 6.5. Show that we can find an affine variety in $\mathbb{A}^2$ isomorphic to $\mathbb{A}^1 \setminus \{0\}$. **Hint:** Look at $xy = 1$.

Definition 6.6. Let $X \subset \mathbb{A}^n$. Define

$$I(X) = \{f \in \Bbbk[z_1, \ldots, z_n] : f \equiv 0 \text{ on every point of } X.\}$$

Definition 6.7. The **affine coordinate ring** of a variety X is

$$A(X) = \mathbb{k}[z_1, \ldots, z_n]/I(X).$$

Definition 6.8. A regular function on $X \subset \mathbb{A}^n$ is a function locally expressible as $f = \frac{g}{h}$ so that g, h are polynomials with $h \neq 0$. (See the same definition last time for a precise description of what locally means.)

Theorem 6.9 (Nullstellensatz)**.** *The ring of functions of an affine variety* $X \subset \mathbb{A}^n$ *is* $A(X)$.

Proof. This may or may not be given later in the course. □

Remark 6.10. Nullstellensatz is a German word which translates roughly to "Zero Places Theorem."

Definition 6.11. Let $Y \subset \mathbb{A}^n$ be an affine variety and X an affine variety. A regular map

$$\begin{aligned} \phi : X &\rightarrow Y \subset \mathbb{A}^n \\ p &\mapsto (f_1(p), \ldots, f_n(p)) \end{aligned}$$

where $f_1, \ldots, f_n$ are regular functions on X.

Remark 6.12. This definition depends on a specific embedding. That is, our definition of a map ϕ included not just the datum of Y but also the inclusion $Y \hookrightarrow \mathbb{A}^n$.

Remark 6.13. The affine coordinate ring $A(X)$ is an invariant of X, up to isomorphism. This follows from the fact that a composition of two regular maps is regular and the pullback of a regular functions along a regular map is regular.

So, isomorphic affine varieties have isomorphic coordinate rings. In fact, $A(X)$ determines X, up to isomorphism.

Remark 6.14. While we'll be sticking to the classical language, we'd like to at least be aware of what happens in the modern theory.

The basic correspondence, at least in the affine case is that affine varieties over an algebraically closed field are in bijection with finitely generated rings over an that algebraically closed field which have no nilpotents.

Here, a nilpotent means a nonzero element so that some power of that element is 0.

Remark 6.15. In the 1950's, Grothendieck decided to get rid of the three hypotheses

(1) finitely generated

(2) nilpotent free
(3) over an algebraically closed field,

and enlarged the rings that we consider as all commutative rings with unit. Then, Grothendieck showed us how to create a category of geometric objects corresponding to arbitrary such rings. These are called **affine schemes**.

6.3. **The Category of Quasi-Projective Varieties.** We have already described the quasi-projective varieties, we will now have to define morphisms between them.

Let's start with what a morphism between projective varieties means.

Definition 6.16. Let $X \subset \mathbb{P}^n$ be a projective variety. We have affine spaces

$$U_i = \{z_i \neq 0\} \subset \mathbb{P}^n,$$

all isomorphic to $\mathbb{A}^n$. Set $X_i = X \cap U_i$. Then, a **regular function** for X is a map of sets $X \to \mathbb{k}$ so that $f|_{X_i}$ is regular.

It's a little annoying that this invokes coordinates. We'd like to next give a coordinate free definition.

Warning 6.17. We've already mentioned this, but a homogeneous polynomial is not a function on projective space. That is, it is only defined up to scaling.

Remark 6.18. Nevertheless, a ratio of two homogeneous polynomials of the same degree does give a well defined function where the denominator is nonzero.

So, definition Definition 6.16 is equivalent to a regular function being locally expressible as a ratio $\frac{G(z)}{H(z)}$ where G, H are homogeneous of the same degree and $H \neq 0$.

Remark 6.19. Unlike in the affine case, we cannot express such functions globally. That is, the word locally is crucial.

Definition 6.20. A **morphism of quasi-projective varieties** or a **regular map** is a map of sets $\phi : X \to Y \subset \mathbb{P}^m$ so that ϕ is given locally by regular functions.

That is, if z_i is a homogeneous coordinate on $\mathbb{P}^n$, and $U_i = \{z_i \neq 0\} \cong \mathbb{A}^n \subset \mathbb{P}^n$, the map

$$\phi^{-1}(U_i) \to U^i \cong \mathbb{A}^n$$

is given by an m-tuple $f_1, \ldots, f_m$ of regular functions.

Equivalently, $\phi : X \to Y \subset \mathbb{P}^m$, with $X \subset \mathbb{P}^n$, is regular if ϕ is given locally by an $(m+1)$-tuple of homogeneous polynomials of the same degree sending

$$p \mapsto [F_0(p), \ldots, F_m(p)] .$$

Remark 6.21 (Historical Remark)**.** A group in the 19th century did not have the modern definition. Instead, it was a subset of GL_n which was closed under multiplication and inversion. In the same way, a manifold was defined as a subset of $\mathbb{R}^n$.

In the 20th century, modern abstract mathematics was invented, and these objects became defined as sets with additional structure. That is, a manifold because sets with a topology which were locally Euclidean. In the 20th century, a variety because something covered by affine varieties. This represented an enlargement of the category. There are varieties built up as affine varieties which are not globally embeddable in any affine or projective space.

Recall we have a bijection between affine varieties X and coordinate rings $A(X)$.

Definition 6.22. Say $X \subset \mathbb{P}^n$ is a projective variety. Then, $X = V(f_1, \ldots, f_k)$. Define

$$I(X) = \{f \in \mathbb{k}[z_0, \ldots, z_n] : \text{ f vanishes on X.}\}.$$

We define the **homogeneous coordinate ring**

$$S(X) = \mathbb{k}[z_0, \ldots, z_n]/I(X).$$

Remark 6.23. The homogeneous coordinate ring is not an invariant of a projective variety. If we look at the rational normal curve

$$\mathbb{P}^1 \to \mathbb{P}^2$$

$$[x_0, x_1] \mapsto \left[x_0^2, x_0x_1, x_1^2\right]$$

is a regular map. This is an embedding, meaning that $\mathbb{P}^1$ is isomorphic to its image in $\mathbb{P}^2$, which is a conic (meaning that it is $V(f)$ for f a quadratic, here $f = w_0w_2 - w_1^2$). Call the image C.

But, the homogeneous coordinate rings

$$S(\mathbb{P}^1) \not\cong S(C).$$

7. 2/4/16

7.1. **Logistics and review.** Homework 1 is due today. It can be submitted on Canvas (as is preferred) or on paper. Homework 2 will be posted today and due Friday February 2/12

Recall the definition of a regular function or regular map on a variety. Suppose we have a map $X \to Y$ with $X \subset \mathbb{P}^m, Y \subset \mathbb{P}^n$. Then, we require that for each pair of affine open subsets, the map is given by some n-tuple of regular functions. That is, using the definition given last time in order to produce a regular map, we would have to produce one on the standard coordinate charts of $\mathbb{P}^n, \mathbb{P}^m$.

Remark 7.1. There is another way of defining a regular map, which is usually easier to specify.

A regular map $\phi : X \to Y$ can be given locally by an $(n+1)$-tuple of homogeneous polynomials on $\mathbb{P}^m$, with no common zeros on X. That is,

$$\begin{aligned} X &\to Y \\ [x_0, \ldots, x_m] &\mapsto [F_0, \ldots, F_m] \end{aligned}$$

where F_i are homogeneous polynomials of the same degree d.

Warning 7.2. You may not find a single tuple of polynomials working everywhere on X. That is, one may have to use different collections of polynomials on different open sets, which agree on the intersections.

7.2. **More on Regular Maps.**

Remark 7.3. As mentioned before, the image of a projective variety under a regular map is again a projective variety. We'll see this in a couple weeks. One way to prove this uses "resultant." (Another way is to use the cancellation theorem, though we won't see this in class.)

Example 7.4 (Plane Conic Curve)**.** Here is an example when we cannot construct a regular map with only a single collection of equations.

Consider

$$C = V(x^2 + y^2 + z^2) \subset \mathbb{P}^2.$$

Once we have the language, this will be called a **plane conic curve**, just because this is the zero locus of a degree two equation in the plane, $\mathbb{P}^2$.

To draw this, look in the standard affine open $U = \{[x, y, z] : z \neq 0\}$. This is just the unit circle with coordinates $x/z, y/z$.

Now, take the top of the circle, which is the point $q := [0, 1, 1]$ in projective coordinates or $(0, 1)$ in affine coordinates. Now, draw the line joining $(0, 1)$ and a point p, and send it to the line $y = 0$. This is known as the **stereographic projection**.

This gives rise to a map

$$\begin{aligned}\pi_1 : C &\to \mathbb{P}^1 \\ [x, y, z] &\mapsto [x, y - z] .\end{aligned}$$

This is well defined, except at the single point $[0, 1, 1]$, where both $x = y - z = 0$.

We can, however, extend this map by constructing different homogeneous polynomials which don't all vanish at q, but do agree on the overlap. We will extend the map π_1 to q by multiplying the map by $y + z$. Explicitly, take

$$\begin{aligned}\pi_2 : C &\to \mathbb{P}^1 \\ [x, y, z] &\mapsto [y + z, -x] .\end{aligned}$$

Note that the common zero locus of the two polynomials $y + z = -x = 0$, which is simply the point $q_2 := [0, -1, 1]$. Let's see why these maps agree on the locus away from q, q_2. On this locus, we have

$$\begin{aligned}[x, y - z] &= [x(y + z), (y - z)(y + z)] \\ &= \left[x(y + z), y^2 - z^2\right] \\ &= \left[x(y + z), -x^2\right] \\ &= [y + z, -x] .\end{aligned}$$

And so the two maps agree away from q, q_2.

So, combining the maps π_1, π_2, we obtain a map $\pi : C \to \mathbb{P}^1$.

Exercise 7.5. Show π is an isomorphism. **Hint:** Check it on an open covering of C by showing π_1 and π_2 are isomorphisms onto their image, where they are defined.

7.3. **Veronese Maps.**

Definition 7.6. Fix n, d. Define the map

$$\begin{aligned}\nu_{d,n} : \mathbb{P}^n &\to \mathbb{P}^N \\ [x] &\mapsto \left[x^I\right]_{\{\#I = d\}} .\end{aligned}$$

Here, I ranges over all multi-indices of degree d and N one less than the number of monomials of degree d in $n + 1$ variables. That is, $N = \binom{n+d}{d} - 1$ by Exercise 7.7.

Exercise 7.7. Show that the number of monomials of degree d in $n+1$ variables is $\binom{n+d}{d}$. **Hint:** Use the "stars and bars" technique by representing a monomial by a collection of $n+d$ slots, where there are d stars corresponding to variables $x_0, \ldots, x_n$, and one places a bar in the d slots corresponding to a dividing line between the x_i variables and x_{i+1} variables.

Example 7.8. Take $n = 2, d = 2$. Then, the 2-Veronese surface is

$$\begin{aligned} \nu_{2,2} : \mathbb{P}^2 &\to \mathbb{P}^5 \\ [x,y,z] &\mapsto \left[x^2, y^2, z^2, xy, xz, yz\right]. \end{aligned}$$

Lemma 7.9. *The image of $\nu_{d,n}$ is a subvariety of $\mathbb{P}^N$ which is the zero locus of the polynomials*

$$\left\{x^I x^J - x^K x^L : I + J = K + L\right\}.$$

Example 7.10. The 2-Veronese surface S, defined as the image of

$$\begin{aligned} \nu_{2,2} : \mathbb{P}^2 &\to \mathbb{P}^5 \\ [x,y,z] &\mapsto \left[x^2, y^2, z^2, xy, xz, yz\right] \end{aligned}$$

is the vanishing locus of

$$\begin{aligned} w_3^2 &= w_0 w_1 \\ w_4^2 &= w_0 w_2 \\ w_5^2 &= w_1 w_2 \\ w_3 w_4 &= w_0 w_5 \\ w_3 w_5 &= w_1 w_4 \\ w_4 w_5 &= w_2 w_3. \end{aligned}$$

In other words, you take two terms and try to express it as a linear combination of two other terms.

Proof of Lemma 7.9.

Question 7.11. Are the equations given above all equations?

Question 7.12. Is the common zero locus equal to the image S?

The answer to both questions is yes, as we will now verify. To see this is all polynomials, consider the map

$$\begin{array}{c} \{\text{ homogeneous quadratic polynomials in } w_0, \ldots, w_5\} \\ \downarrow \\ \{\text{ homogeneous quartic polynomials in } X, Y, Z\}. \end{array} \tag{7.1}$$

This is a surjective map from a 21 dimensional vector space to a 15 dimensional vector space. Its kernel has dimension 6. These are independent because each polynomial has a monomial which appears uniquely in that polynomial.

Exercise 7.13. Verify that the common zero locus of these polynomials is S, completing the proof for the 2-Veronese surface.

Exercise 7.14. Verify that the proof generalizes to $\nu_{d,n}$.

□

Remark 7.15. In fact, the quadratic polynomials given in Lemma 7.9 generate the ideal of the Veronese, meaning that any higher degree polynomial vanishing on the Veronese lies in the ideal generated by the quadrics, though this takes some more work. (One method uses Gröbner bases to count Hilbert polynomials.)

7.4. **Segre Maps.**

Definition 7.16. Consider the map

$$\begin{aligned} \sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n &\to \mathbb{P}^{(m+1)(n+1)-1} \\ ([x], [y]) &\mapsto [\ldots, x_i y_j, \ldots]. \end{aligned}$$

Lemma 7.17. *The image* $\Sigma_{m,n} := \sigma_{m,n}(\mathbb{P}^m \times \mathbb{P}^n) \subset \mathbb{P}^{(m+1)(n+1)-1}$ *is a variety.*

Proof. Consider the quadratic equations

$$\left\{ w_{ij} w_{kl} = w_{kj} w_{il} : 0 \le i, k \le m, 0 \le j, l \le n \right\}.$$

Then, the common zero locus of these is Σ. □

Remark 7.18. The set $\mathbb{P}^m \times \mathbb{P}^n$ is not a priori a variety, since it does not start inside some projective space. However, the Segre map above does realize it as a variety.

Question 7.19. How can one describe a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$? We'll answer this question next time.

8. 2/5/16

8.1. **Overview and Review.** Today, we'll finish our discussion of Veronese and Segre maps. On Monday, we'll be moving on to start chapter 3.

Let's return to the Veronese map and Veronese varieties.

Recall a Veronese map is the map given by

$$\nu = \nu_{d,n} : \mathbb{P}^n \to S \subset \mathbb{P}^N$$
$$[x] \mapsto \left[\ldots x^I \ldots\right]$$

where x^I ranges over all monomials of degree d in $x_0, \ldots, x_n$. The rational normal curve is the case $n = 1$.

The image S is called a Veronese variety. More generally, we say a Veronese map may be given as

$$[x] \mapsto [f_0(x), \ldots, f_N(x)]$$

where $f_0, \ldots, f_n$ is a basis for the vector space of homogeneous polynomials of degree d on $\mathbb{P}^n$. Similarly, the image of such a map is a Veronese Variety.

8.2. **More on the Veronese Variety.**

Remark 8.1. The map ν is an embedding. One way to say this is that there exists a regular map $S \xrightarrow{\phi} \mathbb{P}^n$ so that $\phi \circ \nu = \mathrm{id}$.

Example 8.2. Here is an example of Remark 8.1. Take $n = 1$, so that

$$\nu_{1,n} : [x_0, x_1] \mapsto \left[x_0^d, x_0^{d-1}x_1, \ldots, x_0 x_1^{d-1}, x_1^d\right].$$

Then, the inverse map is

$$\phi_1 : [z_0, \ldots, z_d] \mapsto [z_0, z_1],$$

defined away from $[0, \ldots, 0, 1]$. We can also define

$$\phi_2 : [z_0, \ldots, z_d] \mapsto [z_{d-1}, z_d]$$

which is only defined away from $[1, \ldots, 0]$.

Exercise 8.3. Verify that every point of $\operatorname{im} \nu$ is either contained in the domain of definition of ϕ_1 or ϕ_2, and that ϕ_1 and ϕ_2 agree on the intersection of their domains of definition. Conclude that they glue to give a well defined map $\phi : \mu\nu \to \mathbb{P}^1$.

Then, we can define ϕ by gluing together ϕ_1 and ϕ_2.

Remark 8.4. If $Z \subset \mathbb{P}^n$ is any variety then $\nu(Z) \subset S \subset \mathbb{P}^N$ is a subvariety of $\mathbb{P}^N$. To see this, say $Z \subset \mathbb{P}^n$ is a subvariety with

$$Z = V(F_\alpha(X))$$

with $\deg F_\alpha = d_\alpha$ and F_α are homogeneous.

Exercise 8.5. If $d \geq \max d_\alpha$, we can write Z as the common zero locus of homogeneous polynomials of degree d. *Hint:* Consider

$$\left\{ F_\alpha x^I : x^I \text{ ranges over monomials of degree } d - d_\alpha \right\}.$$

Show that the zero locus of the above set is exactly F. A further hint for this: If $x_0 F, \ldots, x_d F$ all vanish at a point, then F vanishes at that point.

Now, look at the Veronese map

$$\begin{aligned} \nu : \mathbb{P}^n &\to \mathbb{P}^N \\ [x] &\mapsto \left[\ldots, x^I, \ldots\right]. \end{aligned}$$

Call z_i the coordinates of $\mathbb{P}^N$. Then, we have a pullback map

$$\begin{array}{c} \{ \text{ homogeneous polynomials of degree } m \text{ on } \mathbb{P}^N \} \\ \Big\downarrow \nu^* \\ \{ \text{ homogeneous polynomials of degree } md \text{ on } \mathbb{P}^n \} \end{array} \tag{8.1}$$

This map ν^* is surjective. If $Z \subset \mathbb{P}^n$ is any variety, we can assume it is the zero locus of finitely many polynomials. Hence, for some m, we can write Z as the zero locus of polynomials G_α of degree $m \cdot d$. Then, we take polynomials of degree m on $\mathbb{P}^N$ so that the image of these polynomials under ν^* are the polynomials G_α.

Example 8.6. Take

$$\begin{aligned} \nu_{2,2} : \mathbb{P}^2 &\to \mathbb{P}^5 \\ [x, y, z] &\mapsto \left[x^2, y^2, z^2, xy, xz, yz\right]. \end{aligned}$$

Where, the coordinates on $\mathbb{P}^5$ are $w_0, \ldots, w_5$. Take $Z = V(x^3 + y^3 + z^3) \subset \mathbb{P}^2$. We claim $\nu(Z) \subset S \subset \mathbb{P}^5$ is a variety. We can write

$$\begin{aligned} Z &= V(x^3 + y^3 + z^3) \\ &= V(x^4 + y^3 x + z^3 x, x^3 y + y^4, z^3 y, x^3 z + y^3 z + z^4) \\ &= \nu^{-1}\left(V(w_0^2 + w_1 w_3 + w_2 w_4, w_0 w_3 + w_1^2 + w_2 w_5, w_0 w_4 + w_1 w_5 + w_2^2)\right). \end{aligned}$$

The last equality holds because x^2 is the restriction of w_0 under the Veronese map, and so x^4 is w_0^2, etc. Finally, we have

$$\nu(S) = S \cap V\left(w_0^2 + w_1w_3 + w_2w_4, w_0w_3 + w_1^2 + w_2w_5, w_0w_4 + w_1w_5 + w_2^2\right)$$

Then, $\nu(Z) \subset \mathbb{P}^5$ is the zero locus of the above three quadratic polynomials written above and the six quadratic polynomials in $w_0, \ldots, w_5$ that define the surface S.

8.3. **More on the Segre Map.** Recall the **Segre map** is

$$\begin{aligned} \sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n &\to \mathbb{P}^{(m+1)(n+1)-1} \\ ([x], [y]) &\mapsto \left[\ldots x_i y_j \ldots\right]. \end{aligned}$$

The image is a variety in $\mathbb{P}^N$, called **the Segre variety** defined by quadratic polynomials.

Example 8.7. Take $m = n = 1$. We have a map

$$\begin{aligned} \sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^3 \\ ([x_0, x_1], [y_0, y_1]) &\mapsto [x_0y_0, x_0y_1, x_1y_0, x_1y_1]. \end{aligned}$$

In this case, it's very simple to see the image is

$$S = V(w_0w_3 - w_1w_2).$$

In other words, the image is a quadric hypersurface. Recall, a hypersurface is, by definition, the zero locus of a single polynomial in projective space. The degree of a hypersurface is just the degree of that polynomial. This is also called a hyperboloid, over the real numbers. Observe that the fibers of $\mathbb{P}^1 \times \mathbb{P}^1$ are mapped to lines under the Segre embedding. These are the two "rulings of the hyperboloid. See Figure 3.

Remark 8.8. On an intuitive level, isomorphism differs very strongly from homeomorphism. For example, if you're familiar with Riemann surfaces which are of the same genus are homeomorphic (in the Euclidean topology). Two Riemann surfaces are biholomorphic if they are isomorphic as algebraic varieties.

Remark 8.9. We define a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ to be a subvariety of the image of $\sigma = \sigma_{m,n}$, which we call $\Sigma \subset \mathbb{P}^N$. with $N = (m+1)(n+1) - 1$.

FIGURE 3. Hyperboloids "in the wild." From left to right: A hyperboloid model, a hyperboloid at Kobe Port Tower, Kobe, Japan, and cooling hyperbolic towers at Didcot Power Station, UK. These are all examples of quadric surface scrolls with a double ruling.

We have a map

(8.2)

$$\left\{ \text{homogeneous polynomials of degree } d \text{ on } \mathbb{P}^N \right\}$$
$$\Big\downarrow \sigma^*$$
$$\{ \text{bihomogeneous polynomials of bidegree } (d, d) \text{ on } \mathbb{P}^m \times \mathbb{P}^n \}.$$

Then, any variety on $\mathbb{P}^m \times \mathbb{P}^m$ is the zero locus of a polynomial of bidegree d, d on $\mathbb{P}^m \times \mathbb{P}^n$. In fact, we can broaden this to the zero locus of any collection of bihomogeneous polynomials. Here, bihomogeneous means that the polynomials are homogeneous when considered separately in the variables for $\mathbb{P}^m$ and $\mathbb{P}^n$.

Exercise 8.10. Verify that the Segre map is injective.

9. 2/8/16

9.1. **Review.** Recall the Segre map is given by

$$\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \sigma \subset \mathbb{P}^{(m+1)(n+1)-1}$$
$$([x], [y]) \mapsto [\ldots x_i y_j \ldots]$$

with $z_{ij} = x_i y_j$. The image Σ is a variety cut out by

$$\Sigma = V(z_{ij} z_{kl} - z_{il} z_{kj})$$

as i, l, k, j run over all 4 tuples with $0 \leq i, k \leq m, 0 \leq j, l \leq n$.

Example 9.1. If $m = n = 1$, the map is given by

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^3 \\ ([x_0, x_1], [y_0, y_1]) &\mapsto [x_0y_0, x_0y_1, x_1y_0, x_1y_1]. \end{aligned}$$

Take coordinates $z_0, \ldots, z_3$. The image is

$$\Sigma = V(z_0z_3 - z_1z_2).$$

We now just view this $\mathbb{P}^m \times \mathbb{P}^n$ as a variety by viewing it as $\Sigma \subset \mathbb{P}^{(m+1)(n+1)-1}$.

For the rest of the day, for convenience we'll use the notation

$$N := (m+1)(n+1) - 1.$$

9.2. **Even More on Segre Varieties.** Note, if we have a bihomogeneous polynomial in x, y, we get a well defined zero locus.

Definition 9.2. A **subvariety of** $\mathbb{P}^m \times \mathbb{P}^n$ is a subvariety of $\Sigma \subset \mathbb{P}^N$ which are the zero locus of some bihomogeneous polynomials of bidegrees (d_α, d_α).

Exercise 9.3. Show that zero loci of bihomogeneous polynomials (a_α, b_α) (where we do not impose the condition that $a_i = b_i$) form a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$. *Hint:* Say $a_i \neq b_i$. If $a_i < b_i$, then replace that polynomial by its product with all monomials in the first $m+1$ variables of degree $b_i - a_i$.

Example 9.4. Take the twisted cubic

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ t &\mapsto \left[1, t, t^2, t^3\right] \subset V(z_0z_3 - z_1z_2). \end{aligned}$$

That is,

(9.1)
$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 = \Sigma \\ & \searrow \quad \swarrow_{\iota} & \\ & \mathbb{P}^3. & \end{array}$$

In $\mathbb{P}^3$, the twisted cubic is the zero locus of

$$\begin{aligned} z_0z_3 - z_1z_2 \\ z_0z_2 - z_1^2 \\ z_1z_3 - z_2^2. \end{aligned}$$

The latter two polynomials pull back under ι to

$$x_0x_1y_0^2 - x_0^2y_1^2$$
$$x_0x_1y_1^2 - x_1^2y_0^2.$$

Now, note that the polynomials factor as

$$x_0(x_1y_0^2 - x_0y_1^2)$$
$$-x_1(x_1y_0^2 - x_0y_1^2).$$

So, the common zero locus of these two polynomials is precisely that of

$$x_1y_0^2 - x_0y_1^2.$$

Then, we see the maps in Equation 9.1 are given by

$$\begin{array}{ccc} t & \longrightarrow & \left(\left[1, t^2\right], [1, t]\right) \\ & \searrow \quad \swarrow & \\ & \left[1, t, t^2, t^3\right]. & \end{array} \tag{9.2}$$

Lemma 9.5. *Suppose* $X \subset \mathbb{P}^m$ *and* $Y \subset \mathbb{P}^n$ *are projective varieties. Then, their product* $X \times Y \subset \mathbb{P}^m \times \mathbb{P}^n$ *is a projective variety.*

Proof. To get the equations cutting out X in $\mathbb{P}^m$, those cutting out Y in $\mathbb{P}^n$ and those cutting out the Segre variety $\Sigma_{m,n}$. □

Fact 9.6. Suppose $f : X \to Y$ is a regular map. Then, the graph of f, which is the set of $(x, f(x))$ with $x \in X$, is a subvariety $\Gamma_f \subset X \times Y$.

As an ideal of how to prove this, one can work in local coordinates, in which the map $X \to Y$ is locally given by some polynomials.

9.3. **Cones.**

Definition 9.7. Suppose $X \subset \mathbb{P}^n$ and $p \in \mathbb{P}^n$. Then, we define the **cone over** X **with vertex** p

$$\overline{X} := \overline{p, X} = \cup_{q \in X} \overline{pq}.$$

Here, $\overline{X}$ implicitly depends on p.

Remark 9.8. If we work in affine space, and take the point p at infinity, the cone over X with vertex p becomes a cylinder in that affine chart.

Lemma 9.9. *Suppose $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ is a hyperplane. Let $p \in \mathbb{P}^n \setminus \mathbb{P}^{n-1}$. Let $X \subset \mathbb{P}^{n-1}$ be any variety.*

Such a cone is a projective variety.

Proof. We can assume $\mathbb{P}^{n-1} = V(Z_n)$ and take $p = [0, \ldots, 0, 1]$. If

$$X = V(F_\alpha(z_0, \ldots, z_{n-1}))$$

then

$$\overline{X} = V(F_\alpha(z_0, \ldots, z_n)).$$

Here X is viewed as the zero locus of polynomials in the first n variables while $\overline{X}$ is viewed in all $n+1$ variables.

To see this is in fact the cone, we argue as follows. Take $q = [z_0, \ldots, z_{n-1}, 0] \in X$. If we look at the line $\overline{p}q$, an open set of which consists of points of the form

$$[z_0, \ldots, z_{n-1}, *] .$$

Then, a polynomial in the first n variables vanishes on such a point, if and only if it vanishes on $q \in X$. □

Remark 9.10. In fact, a converse holds as well: If we have $Y \subset \mathbb{P}^n$ and can find a hyperplane so that all equations of Y are defined in the variables of the hyperplane, then Y is a cone.

Definition 9.11. We say $\mathbb{P}^k, \mathbb{P}^l \subset \mathbb{P}^n$ are **complementary** if we can write

$$\mathbb{P}^n = \mathbb{P}V$$
$$\mathbb{P}^k = \mathbb{P}W$$
$$\mathbb{P}^l = \mathbb{P}U$$

with $V = W \oplus U$.

Exercise 9.12. Let $\mathbb{P}^k \mathbb{P}^l \subset \mathbb{P}^n$. Show the following are equivalent:

(1) $\mathbb{P}^k$ and $\mathbb{P}^l$ are complementary
(2) $\mathbb{P}^k$ are disjoint and span $\mathbb{P}^n$
(3) $\mathbb{P}^k$ and $\mathbb{P}^l$ are disjoint and $k + l = n - 1$.

We now make a more general definition of cones.

Definition 9.13. Suppose $\mathbb{P}^k$ and $\mathbb{P}^l$ are complementary in $\mathbb{P}^n$. Let $X \subset \mathbb{P}^k$. A **cone over** X is $\overline{X, \mathbb{P}^l} = \cup_{q \in X} \overline{q, \mathbb{P}^l}$.

Exercise 9.14. Show that a cone as defined in Definition 9.13 is a variety. *Hint:* Show that such a cone can be viewed as taking $l+1$ cones over X with the $l+1$ points being the $l+1$ coordinate points of $\mathbb{P}^l$, and iteratively apply Lemma 11.3.

9.4. **Cones and Quadrics.** Now, for this subsection, suppose we have a field k, and the characteristic of $\mathbb{k}$ is not 2.

Definition 9.15. A **quadric** $X \subset \mathbb{P}^n = \mathbb{P}V$ with $V \cong \mathbb{k}^{n+1}$ is the zero locus of a single homogeneous quadric polynomial Q.

Remark 9.16. We have a one to one correspondence between

$$\begin{array}{c} \{\text{ homogeneous quadratic polynomials } Q \text{ on } V\,\} \\ \Big\downarrow \\ \{\text{ symmetric bilinear forms } Q_0 : V \times V \to \mathbb{k}\}. \end{array} \tag{9.3}$$

Here, $Q(v) = Q_0(v, v)$ and $Q_0(v, w) = \frac{Q(v+w)-Q(v)-Q(w)}{2}$.

Fact 9.17. Any symmetric bilinear form can be diagonalized. That is, there exist coordinates on V so that

$$Q(x_0, \ldots, x_n) = \sum_{i=1}^{k} x_i^2.$$

Further, up to change of coordinates, this number k is uniquely determined.

Definition 9.18. Let Q be a quadratic form. We define the **rank of** Q to be the integer k, defined in Fact 9.17.

Exercise 9.19. Show that the rank of Q is equal to the rank of the linear operator

$$\begin{aligned} \hat{Q} : V &\to V^\vee \\ v &\mapsto Q(v, \bullet). \end{aligned}$$

Example 9.20. In $\mathbb{P}^1$ there is a quadric of the form $x_0^2 + x_1^2$, cutting out two points, and the "double point" x_0^2.

Example 9.21. Over $\mathbb{P}^2$, the rank 3 case is $x_0^2 + x_1^2 + x_2^2$. This is a smooth conic, and conversely any quadratic hypersurface which is not a union of lines is equivalent to this one. For example, this is projectively equivalent to $x_0x_2 - x_1^2$. The rank 2 case is $x_0^2 - x_1^2$, which is a cone over two points, or a union of two lines. The rank 1 case is x_0^2, which is a "double line."

Example 9.22. In $\mathbb{P}^3$ we have four quadrics, up to projective equivalence. We first have the rank 4 case, $x_0^2 + x_1^2 + x_2^2 + x_3^2$, which is a smooth quadric hypersurface. For example, this is projectively equivalent $x_0 x_3 - x_1 x_2$. Next, we have the rank 3 case $x_0^2 + x_1^2 + x_2^2$, which is a quadric cone. The rank 2 case $x_0^2 + x_1^2$, which is a union of two hyperplanes. Finally, we have the rank 1 case, which is x_0^2, which is a "double plane."

Remark 9.23. You may have been taught back in school that there are lots of quadrics. For example

(1) spheres
(2) ellipsoids
(3) hyperboloids.

But, here, we're saying a quadric is swept out by two families of lines. How would you find the lines on a sphere? A line on the surface will be the intersection of a line with its tangent plane. Where would this appear on the sphere? Say it is $V(x^2 + y^2 + z^2 - 1$. Then, this will factor as $x^2 + y^2 = (x + iy)(x - iy)$, and this factors as two complex lines. However, we don't see these over the real numbers.

10. 2/10/15

10.1. **Review.** Recall, a quadric $Q \subset \mathbb{P}^n$ is the zero locus of a single homogeneous polynomial $Q(x)$. Over an algebraically closed field of characteristic not equal to 2, such a quadric is characterized by its rank.

Example 10.1. In $\mathbb{P}^3$, there are four quadrics up to isomorphism, classified by rank. This follows from an analysis of quadratic forms in four variables, which is a standard result from linear algebra.

(1) Rank 4: $V(x^2 + y^2 + z^2 + w^2)$. This is the smooth doubly ruled paraboloid we usually draw. Note that this is projectively equivalent to $V(xy - zw)$, which one may realize as the image of the segre map, $\Sigma_{1,1} \cong \mathbb{P}^1 \times \mathbb{P}^1$.
(2) Rank 3: $V(x^2 + y^2 + z^2)$. This is a cone over a plane conic.
(3) Rank 2: $V(x^2 + y^2)$ factors as a product of linear forms, and so it is a union of two planes.
(4) Rank 1: $V(x^2)$ is a double plane.

10.2. **More on Quadrics.**

Example 10.2. This is different if one works over the real numbers or in affine space. For example, rank 4 quadrics in $\mathbb{P}^3_{\mathbb{R}}$, over the real numbers, we have three different kinds. The reason is that quadrics over the reals are determined by rank and signature. So, representatives for these three isomorphism classes are

(1) $V(x^2 + y^2 + z^2 + w^2)$, there are no points in $\mathbb{P}^3_{\mathbb{R}}$ on this
(2) $V(x^2 + y^2 + z^2 - w^2)$, this is a sphere
(3) $V(x^2 + y^2 - z^2 - w^2)$, this is the hyperboloid

Example 10.3. Going one step further, let's look at quadrics over $\mathbb{R}$, as described in the above example, when we restrict to an affine chart in $\mathbb{A}^3$. In the first case, $V(x^2 + y^2 + z^2 + w^2)$ this is empty, so it will look the same on every affine chart $\mathbb{A}^3$. For the second case, let's look at how the sphere meets the plane at ∞. There are three cases

(1) If it does not meet the plane at infinity, it will be a sphere
(2) if it meets the plane at infinity along a curve it will be a "hyperboloid of two sheets," which looks kind of like a hyperboloid going up and a separate hyperboloid going down. This is the surface you get by taking a vertically oriented hyperbola in the plane, and rotating around the y axis.
(3) If it meets the plane at infinity tangentially at a point, it will be a single hyperboloid.

For the third case, a hyperboloid will either meet the plane at infinity in

(1) a smooth conic
(2) a conic which is a union of two lines.

Remark 10.4. Life is so much easier over the complex numbers! See the difference between Example 10.2 and Example 10.3.

10.3. **Projection Away from a Point.**

Definition 10.5. Start with a hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$. We can define a map

$$\begin{aligned} \mathbb{P}^n \setminus \{p\} &\to H \\ q &\mapsto \overline{qp} \cap H. \end{aligned}$$

This map is called **projection away from the point** p **onto** H.

Remark 10.6. This is how projective space got its name.

Exercise 10.7. Choose homogeneous coordinates z on $\mathbb{P}^n$ and write $H = V(Z_n)$ and

$$p = [0, \ldots, 0, 1].$$

Then,

$$\pi_p : [z_0, \ldots, z_n] \mapsto [z_0, \ldots, z_{n-1}]$$

is projection away from p onto H.

Remark 10.8. We can also think of this in terms of vector spaces. H corresponds to an n-dimensional subspace of an $n+1$ dimensional subspace. The points p and q correspond to one dimensional subspaces, and the line through them is the two dimensional space spanned by the two corresponding one dimensional spaces. This two dimensional space necessarily meets the n dimensional subspace corresponding to H.

Theorem 10.9. *If $X \subset \mathbb{P}^n$ is any projective variety and $p \notin X$, then $\overline{X} = \pi_p(X) \subset \mathbb{P}^{n-1}$ is a projective variety.*

We will delay the proof until we discuss resultants. This theorem is also the key step in showing that the image of a variety under a regular map is a variety.

10.4. **Resultants.** The motivating question for resultants is the following.

Question 10.10. Let $\mathbb{k}$ be a field and let

$$\begin{aligned} f(x) &= a_m x^m + \cdots + a_0 \\ g(x) &= b_n x^n + \cdots + b_0 \end{aligned}$$

When to f and g have a common zero?

We can ask a similar question for homogeneous polynomials, which is almost the same, although is slightly different.

Question 10.11. Suppose

$$\begin{aligned} F(X, Y) &= a_m x^m + a_{m-1} x^{m-1} y + \cdots + a_0 y^m \\ G(X, Y) &= b_n x^n + \cdots + b_0 y^n. \end{aligned}$$

When do F, G have a common zero in $\mathbb{P}^1$.

These are almost equivalent, except for technical bookkeeping details, so we'll just look at Question 10.10, for simplicity.

Here is a restatement of Question 10.10.

Lemma 10.12. *Let $\mathbb{P}^m$ be the space of polynomials of degree m on $\mathbb{P}^1$. This is isomorphic to $\mathbb{P}^m$ by writing the polynomials as*

$$a_m x^m + \cdots + a_0,$$

and letting the coordinates of $\mathbb{P}^m$ *be the variables* $a_m, \ldots, a_0$*. Now, consider*

$$\Sigma = \{(f, g) \in \mathbb{P}^m \times \mathbb{P}^n : f \text{ and } g \text{ have a common zero}\}.$$

The variety Σ *is a subvariety of* $\mathbb{P}^m \times \mathbb{P}^n$*.*

To prove this, we will find explicit defining equations for Σ

Proof. Define V_m to be the vector space of polynomials of degree at most m. For fixed polynomials f and g, consider the map

$$\begin{aligned} \phi_{f,g} : V_{n-1} \oplus V_{m-1} &\to V_{m+n-1} \\ (A, B) &\mapsto fA + gB. \end{aligned}$$

Lemma 10.13. *The map* $\phi_{f,g}$ *is an isomorphism if and only if* f *and* g *have no common zero.*

Proof. Note that the source and target of this map both have dimension $m + n$. So, showing this is an isomorphism is equivalent to showing it is injective or surjective.

First, if f and g have a common zero at p, then the image of this map is contained the the subset of polynomials of degree at most $m + n - 1$ vanishing at p.

So, we only need verify the converse. If this $\phi_{f,g}$ fails to be an isomorphism, then this map has a kernel. Say $fA + gB \in \ker \phi_{f,g}$. This means $fA + gB = 0$ as a polynomial. This means that wherever the first term vanishes, the second term must vanish. So, gB vanishes at the m roots of f. This implies that g vanishes at at least one root of f, since B can only account for $m - 1$ of the roots of f. □

Now, we will write out the matrix representation of ϕ. For this, we'll take a simple basis, given by powers of x. That is, for V_{n+m-1}, take the basis $1, x, x^2, \ldots, x^{n+m-1}$. and for $V_{n-1} \oplus V_{m-1}$, take a basis given by

$$(1, 0), (x, 0), \ldots, (x^{n-1}, 0), (0, 1), (0, x), \ldots, (0, x^{m-1})$$

Then, the map is given by the following matrix (where the columns are not perfectly aligned for general values of m and n, but the idea is that the first row of a_i's and middle row of b_i's get translated right one row at a time).

$$M_{f,g} := \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_m & 0 & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{m-1} & a_m & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & b_{n-1} & b_n & 0 & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & b_{n-1} & b_n & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_n \end{pmatrix}$$

We have that f and g have a common zero if and only if $\deg M = 0$. □

Definition 10.14. The **resultant** of f and g, denoted $R(f, g)$ is $\det M_{f,g}$, where $M_{f,g}$ is defined in the proof of Lemma 10.13.

Remark 10.15. The Euclidean algorithm is essentially a way of row reducing the matrix $M_{f,g}$ yielding the resultant, although we won't make precise how this works.

We would now like to come back and prove Theorem 10.9, using Lemma 10.13, although we won't have time to do this today.

For now, here is one generalization.

Remark 10.16. Let F, G be homogeneous polynomials in $z_0, \ldots, z_r$. We can view F and G as polynomials in z_r with coefficients in $\mathbb{k}[z_0, \ldots, z_{r-1}]$. That is, we would write

$$f(z) = a_m(z_0, \ldots, z_{r-1})z_r^m + \cdots a_0(z_0, \ldots, z_{r-1}).$$

where $a_i(z_0, \ldots, z_{r-1}$ is a homogeneous polynomial of degree $m - i$. We can now form the same matrix $M_{f,g}$ as in the proof of Lemma 10.13, which are now considered as a matrix of polynomials, and this is called $\mathrm{Res}_{Z_r}(F, G)$.

11. 2/12/16

11.1. **Logistics.**

(1) Today, we'll finish chapter 3
(2) There is no class on Monday
(3) Next Wednesday, we'll go through chapter 4
(4) On Friday, we'll start on chapter 5

Chapter 4 is about families of varieties. There isn't much logical content. It's mostly just a definition and some examples.

On Friday, we'll pay off some IOU's and prove some of the Theorem's Joe's been asserting. We'll also get back to proving such theorems today.

11.2. **Review.** Let's start by recalling the basic definition of the resultant.

Write

$$\begin{aligned} f(x) &= a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0 \\ g(x) &= b_n x^n + \cdots + b_0. \end{aligned}$$

Then, recall the resultant is the determinant of the matrix

$$R(f,g) = \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_m & 0 & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{m-1} & a_m & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & b_{n-1} & b_n & 0 & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & b_{n-1} & b_n & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_n \end{pmatrix}$$

Observe that f, g have a common 0 if and only if $R(f, g) = 0$. Observe further that $R(f, g)$ is bihomogeneous of bidegree (m, n) in the a_i and b_j.

Let's next recall the slight generalization of this, mentioned at the end of last class.

Say $f, g \in S[x]$, for S an arbitrary ring. Then, we can still take the resultant $R(f, g)$ as above. We will care especially about this when $R = \Bbbk[z_0, \ldots, z_r]$ and $S[x] \cong \Bbbk[z_0, \ldots, z_{r-1}][z_r]$. Then, we define the notation $R_{z_r}(f, g)$ to be the resultant of $f, g \in \Bbbk[z_0, \ldots, z_r]$ with respect to the last variable.

Recall also the construction of projection away from a point. Take a point $p \in \mathbb{P}^\alpha$ and $H \cong \mathbb{P}^{\alpha-1} \subset \mathbb{P}^\alpha$ a hyperplane, so that $p \notin H$. Then, define

$$\begin{aligned} \pi_p : \mathbb{P}^\alpha \setminus \{p\} &\to \mathbb{P}^{\alpha-1} \\ q &\mapsto \overline{qp} \cap H. \end{aligned}$$

We may as well choose coordinates so that we have

$$p = [0, \ldots, 0, 1]$$
$$H = V(z_\alpha).$$

Then, in these coordinates, the map π_p is

$$[z_0, \ldots, z_\alpha] \mapsto [z_0, \ldots, z_{\alpha-1}].$$

11.3. **Projection of a Variety is a Variety.**

Lemma 11.1. *Let* $X \subset \mathbb{P}^\alpha$ *be a projective variety. Suppose* $p \notin X$*. Then,* $\pi_p(X) = \overline{X} \subset \mathbb{P}^{\alpha-1}$.

Proof. The proof amounts to verifying that

$$\overline{X} = V(\{\mathrm{Res}_{z_\alpha}(f, g) : f, g \in I(X)\}) \subset \mathbb{P}^{\alpha-1},$$

where $\overline{X}$ is the cone over X. To verify this, note that any $r \in \pi_p(X)$, the line $\ell := \overline{p}r$ satisfies $\ell \cap X \neq \emptyset$. This is the same as saying that any pair $f, g \in I(X)$ have some common zero on ℓ. □

Remark 11.2. In Lemma 11.1, we cannot get away with just taking generators of $I(X)$. We will actually need to take all elements of the ideal. If we only took only resultants of generators, it may be that every pair of these generators have a common zero locus, even though all of these have no common zero locus.

Hence, this calculation is not computationally effective. One can make it effective using Gröbner bases, which is indeed carried out in the algebraic geometry computer language Macaulay.

11.4. **Any cone over a variety is a variety.** We'll now move back to projective space of dimension n, instead of α, being written as $\mathbb{P}^n$.

Lemma 11.3. *Let* $p \in \mathbb{P}^n$ *and* $X \subset \mathbb{P}^n$ *a projective variety. Then, set* $\overline{p, X} = \cup_{q \in X} \overline{pq}$.

Remark 11.4. We already did this when X is contained in a hyperplane complementary to p.

Proof. Choose any hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ not containing p. Observe that the cone $\overline{p, \pi_p(X)}$ is the same as the cone $\overline{p, X}$. Both are just the union of lines through p and a point on X. By Lemma 11.1, $\pi_p(X)$ is a variety if X is. But now, we saw last time that the latter is a variety, as its equations are the same as those of $\pi_p(X)$, viewed as equations in one more variable, after a suitable change of coordinates. □

11.5. **Projection of a Variety from a Product is a Variety.**

Lemma 11.5. *Suppose* Y *is a projective variety and* $X \subset Y \times \mathbb{P}^1$*, a projective variety. Let* $\pi : Y \times \mathbb{P}^1 \times Y$ *be the projection. Then,* $\pi(X) \subset Y$ *is a variety.*

Proof. Take all pairs $F, G \in I(X$ and take the resultants with respect to the $\mathbb{P}^1$ factors. □

Lemma 11.6. *Suppose* $X \subset Y \times \mathbb{P}^n$ *be a projective variety and* $\pi : Y \times \mathbb{P}^n \to Y$*. Then,* $\pi(X) \subset Y$ *is a projective variety.*

Proof. The idea is to use induction on n. The base case is Lemma 11.6. We will choose a point $p \in \mathbb{P}^n$ and a hyperplane $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$. We consider the projection map

$$\eta := \mathrm{id} \times \pi_p : Y \times (\mathbb{P}^n \setminus \{p\}) \to Y \times \mathbb{P}^{n-1}$$

So, by induction, it suffices to show the following lemma.

Lemma 11.7. *With notation as above,* $\eta(X) \subset Y \times \mathbb{P}^{n-1}$ *is a projective variety.*

Proof. We'd like to just say the image of a projective variety is a projective variety. The only problem is that the variety can't contain the point. However, it may be that the cross section p intersects the variety X. This works away from the locus of points in Y where p intersects the fiber of X.

That is, define

$$V = \{q \in Y : (q, p) \in X\}.$$

Remark 11.8. There is a typo in the textbook where V is incorrectly defined as the complement of the V defined here.

This is a closed subvariety of X. We just restrict the equations defining X to $Y \times \{p\}$.

Note that $\pi(X) \cap Y \setminus V$ is closed in $Y \setminus V$ and $\pi(X) \supset V$. These together imply that $\pi(X)$ is closed in Y, but straightforward point set topology. □

□

11.6. **The image of a regular map is a Variety.**

Proposition 11.9. *Suppose* $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^n$ *are projective varieties and* $f : X \to Y$ *is a regular map. Then,* $f(X) \subset Y$ *is a projective variety.*

Proof. Locally on X, the map f may be given as a tuple of homogeneous polynomials. That is, f is locally given by

$$[z_0, \ldots, z_m] \mapsto [f_0(z), \ldots, f_n(z)].$$

We start by showing that the graph of a map is closed. Define

$$\Gamma_f := \{(x, y) : x \in X, y \in Y, y = f(x)\} \subset X \times Y.$$

Then, Γ_f is the zero locus of the bihomogeneous polynomials

$$w_i f_j(z) - w_j f_i(z),$$

where w_i are coordinates on Y and z_j are coordinates on X. That is,

$$\Gamma_f \subset X \times Y \subset \mathbb{P}^m \times \mathbb{P}^n$$

is a projective variety.

Finally, by Lemma 11.6, we obtain $\pi(\Gamma_f) = f(X) \subset Y$ is a projective variety. □

Remark 11.10. This proof works over arbitrary field, but the resultant only detects common factors, meaning common points over the algebraic closure (these are called "geometric points").

Remark 11.11. Read the part in the textbook on constructible sets, but don't worry too much about it.

Note that the image of a quasiprojective variety need not be closed. It is true that the image of a constructible subset is constructible, although we won't need this much in the class, it's just good to know.

Remark 11.12. There is a technique for making this process algorithmically efficient.

This is about choosing the right set of generators. If you do this, you don't have to worry about taking all pairs of elements in the ideal, just elements of a Gröbner basis.

For example, see any one of

(1) Brendan Hassett's Introduction to Algebraic Geometry
(2) Chapter 2 of Cox Little O'Shea's algebraic geometry book
(3) Chapter 15 of Eisenbud's algebraic geometry textbook

Gröbner bases are also useful for analyzing degenerations or limits of varieties in families.

12. 2/17/16

12.1. **Overview.** Today, we'll discuss chapter 4. On Friday, we'll start chapter 5.

Remark 12.1. Chapter 4 is a little anomalous. Chapter 4 deals with a very central theme in algebraic geometry, and just gives some examples. The idea is that an algebraic variety is defined by some finite collection of polynomials. These polynomials have finitely many coefficients. So, we can specify a variety by a finite amount of data. So, a set of varieties of a given type typically correspond to points of another variety, which is a parameter space.

This contrasts sharply with differential geometry. For example, it's much harder to give "submanifolds of $\mathbb{R}^2$" the space of a manifold. There are just too many of them. There's also not a way to give them a nice compactification.

However, the set of quadratic curves in $\mathbb{P}^2$ is specified by 6 coefficient. So, the set of conics corresponds to six tuples of coefficients up to scalars, which is isomorphic to $\mathbb{P}^5$.

12.2. **Families.**

Definition 12.2. A **family of varieties in $\mathbb{P}^n$ parameterized by a variety** B a closed subvariety $\mathcal{X} \subset B \times \mathbb{P}^n$.

A **member of a family** $\mathcal{X} \subset B \times \mathbb{P}^n \xrightarrow{\pi} B$ is a fiber $X_b = \pi^{-1}(b) \subset \mathbb{P}^n$.

Example 12.3 (Stupid example)**.** Take $B = \mathbb{P}^1$. Then, consider a point in $B \times \mathbb{P}^n$. This is a family by our definition, but it's fairly stupid, and doesn't look much like a family, because all of the members, but one, are empty, and one member is a point. (The magic word you want is that families are "flat," we won't use this word in this course.)

12.3. **Universal Families.**

Example 12.4 (The universal family of conics)**.** First, recall that a conic curve $C \subset \mathbb{P}^2$ is the zero locus of a single homogeneous polynomial on $\mathbb{P}^2$.

Let V be the vector space of homogeneous quadratic polynomials on $\mathbb{P}^2$. The space of conics is $\mathbb{P}V \cong \mathbb{P}^5$, determined by the five coefficients of a homogeneous polynomial.

We have a family

$$\mathcal{C} = V\left(ax^2 + by^2 + cz^2 + dxy + exz + fyz\right) \subset \mathbb{P}^5_{[a,b,c,d,e,f]} \times \mathbb{P}^2_{[x,y,z]}$$

The family is a family over $\mathbb{P}^5$ by the projection onto the first coordinate

$$\begin{array}{c} \mathcal{C} \\ \downarrow \\ \mathbb{P}^5. \end{array} \tag{12.1}$$

Remark 12.5. The above example Example 12.4 implicitly assumes that two quadratic polynomials determine the same zero locus if and only if they are scalar multiples of each other.

This follows immediately from the Nullstellensatz. We will see this in a few weeks.

However, you can also do this directly. One method to do it directly would be to take five "suitable" points on the conic, and see there is a unique conic up to scaling, passing through them, so it must be that conic.

Example 12.6 (The universal family of degree d curves in $\mathbb{P}^2$)**.** We can generalize Example 12.4 by replacing conics by degree d polynomials.

Take

$$\mathcal{C} = V\left(\sum_{i+j+k=d} a_{ijk}x^iy^jz^k\right) \subset \times\mathbb{P}^N_{[a_{ijk}]} \times \mathbb{P}^2_{x,y,z}.$$

Here $\mathcal{C}$ is viewed as a family over $\mathbb{P}^N$ with $N = \binom{d+2}{2} - 2$. This is the family of all degree d curves in $\mathbb{P}^2$.

Definition 12.7. A **hypersurface** $X \subset \mathbb{P}^n$ **of degree** d is the zero locus of single homogeneous polynomial $F(x_0, \ldots, x_n)$ of degree d.

Example 12.8 (The universal family of hypersurfaces)**.** We can further generalize example Example 12.6 as follows. We have a family

$$\mathcal{X} = V\left(\sum_I a_I x^I\right) \subset \mathbb{P}^N_a \times \mathbb{P}^n_x$$

with

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathbb{P}^N \end{array} \tag{12.2}$$

with $N = \binom{n+d}{d} - 1$. This $\mathcal{X}$ is called the **universal hypersurface** of degree d in $\mathbb{P}^n$.

In the case $n = 2, d = 2$, it is called the **universal conic**.

Remark 12.9. The "universal family" is universal, loosely in the sense that any family of such hypersurfaces has a map to the universal family. (This gets into Hilbert schemes, which we won't discuss.)

Example 12.10 (The universal hyperplane)**.** Define

$$\mathcal{H} = V(a_0x_0 + \cdots + a_nx_n) \subset \mathbb{P}^n_a \times \mathbb{P}^n_x.$$

To avoid confusion, we define $(\mathbb{P}^n)^\vee := \mathbb{P}^n_a$. Abstractly $(\mathbb{P}^n)^\vee \cong \mathbb{P}^n$. It is really just defined as a variety isomorphic to $\mathbb{P}^n$, together with the datum of $\mathcal{H}$ inside the product $\mathcal{H} \subset (\mathbb{P}^n)^\vee \times \mathbb{P}^n$.

If $\mathbb{P}^n \cong \mathbb{P}V$ then $(\mathbb{P}^n)^\vee \cong \mathbb{P}V^\vee$.

Example 12.11. Another special case is when $n = 1$. In this case, we get a universal family

$$\mathcal{D} = V\left(a_0x^d + a_zx^{d-1}y + \cdots + a_dy^d\right) \subset \mathbb{P}^d_a \times \mathbb{P}^1.$$

12.4. **Sections of Universal Families.**

Question 12.12. Given a family $\mathcal{X} \xrightarrow{\pi} B$ with

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & B \times \mathbb{P}^n \\ & \searrow & \downarrow \\ & & B \end{array} \tag{12.3}$$

we can ask, does there exist a section $\mathcal{X} \xleftarrow{\sigma} B$. That is, such a map σ with $\pi \circ \sigma = \mathrm{id}$.

Intuitively, if our base B is a curve, this is just asking whether we can find an algebraic curve inside the family mapping bijectively down to B.

Example 12.13. For example, if we take a quadratic, $ax^2 + bxy + cy^2 = 0$, we can ask whether there exist polynomial functions $x = x(a, b, c), y = y(a, b, c)$ with $ax^2 + bxy + cy^2 = 0$. Then, take

$$x/y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The cases of degree 3 and 4 give the cubic and quartic formulas.

Example 12.14. Similarly, we can ask whether there exists a polynomial function

$$x = x(a, \ldots, f)$$
$$y = y(a, \ldots, f)$$
$$z = z(a, \ldots, f).$$

so that

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0.$$

There will be a whole plane conic curve over each choice of point (x, y, z) in $\mathbb{P}^2$. That is, we're asking for a solution to a quadratic polynomial in more than one variable. The answer ends up being "no," although this is nontrivial to prove.

12.5. **More examples of Families.**

Example 12.15 (The Grassmannian)**.** Let's consider the set of k planes $\Lambda \cong \mathbb{P}^k \subset \mathbb{P}^n$. This is the Grassmannian $G(k+1, n+1)$, the space of $k+1$-dimensional subspaces in an $n+1$-dimensional vector space.

We'll see later that this is a variety, not just a set, and we'll see that the k planes over the Grassmannian in fact form a closed subvariety of $\mathbb{P}^n \times G(k+1, n+1)$.

Example 12.16 (Twisted Cubics)**.** We can try to parameterize the set of twisted cubics $C \subset \mathbb{P}^3$. This is harder than the case of hypersurfaces because it is not just given by one polynomial.

One can try to prove it is the image of a family by writing it as the image of degree 3 maps from $\mathbb{P}^1$.

That is, one can say a twisted cubic is given by a map

$$\mathbb{P}^1 \to \mathbb{P}^3$$
$$[x_0, x_1] \mapsto [f_0(x), \ldots, f_3].$$

The problem is that one can reparameterize the source $\mathbb{P}^1$ curve, because you can reparameterize the curve, and get the same twisted cubic.

Another way to describe a twisted cubic is as a 3 dimensional subspace of the space of homogeneous quadratic polynomials on $\mathbb{P}^3$. That is, we can take $C = V(Q_1, Q_2, Q_3) = V(\Lambda)$. where $\Lambda \subset \{\text{ homogeneous polynomials on } \mathbb{P}^3\} \cong \mathbb{k}^{10}$.

This is the first hint of the construction of the Hilbert scheme, parameterizing projective varieties in great generality.

13. 2/19/16

Remark 13.1. The discussion today implicitly uses the notion of irreducibility. That is, we will use that in the following examples, the complement of a closed subvariety is dense, which holds because the ambient varieties these closed subvarieties these lie in (which will usually be some product of projective spaces) are irreducible.

13.1. Review.

Definition 13.2. A **family of projective varieties parameterized by a variety** B is a closed subset

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & B \times \mathbb{P}^n \\ & \searrow & \downarrow \\ & & B \end{array} \tag{13.1}$$

A **member of this family** is a fiber $X_b := \pi^{-1}(b)$ for $b \in B$.

As this stands, this variety does not have enough structure, because the members of this family will currently have almost nothing to do with one another. The dimension can jump, some fibers can be empty, etc. We'd like for families to have members looking similar to one another (the technical term being flat) but we won't go in to this more in this class.

13.2. Generality.

Definition 13.3. Given a family

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & B \times \mathbb{P}^n \\ & \searrow & \downarrow \\ & & B \end{array} \tag{13.2}$$

A **general member** of the family has a property $\mathbb{P}$ if the set

$$\{b \in B : X_b \text{ has property } \mathbb{P}\} \subset B$$

contains an open dense subset of B.

Let's see some examples of generality.

Example 13.4. A general triple of points

$$(p_1, p_2, p_3) \in (\mathbb{P}^2)^3$$

is not collinear. In more precise language, take the family over the base $(\mathbb{P}^2)^3$ given by

$$\mathcal{X} := \{((p, q, r), s) : s \in \{p, q, r\}\} \subset \left(\mathbb{P}^2\right)^3 \times \mathbb{P}^2.$$

To check this is a family, we first have to say this locus is a closed subset.

Lemma 13.5. *We have $\mathcal{X} \subset \left(\mathbb{P}^2\right)^3 \times \mathbb{P}^2$ is a subvariety.*

Proof. Recall the diagonal $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the set of pairs (p, p) with $p \in \mathbb{P}^2$. This is indeed a closed subvariety. Let $\pi_{ij} : \mathcal{X} \to \mathbb{P}^2 \times \mathbb{P}^2$ be the projection to the i and jth copies of $\mathbb{P}^2$. Then,

$$\mathcal{X} = \pi_{14}^{-1}(\Delta) \cup \pi_{24}^{-1}(\Delta) \cup \pi_{34}^{-1}(\Delta).$$

Now, we can finally make sense of the statement that a general triple of points p_1, p_2, p_3 in $\mathbb{P}^2$ are not collinear.

This simply means that the locus of points

$$\left\{(p, q, r) \in \left(\mathbb{P}^2\right)^3 : p, q, r \text{ are collinear}\right\} \subset (\mathbb{P}^2)^3$$

is contained in a closed subset of $\left(\mathbb{P}^2\right)^3$.

This is equivalent to not being dense because the complement of a closed subset is open, and every open set in $(\mathbb{P}^2)^3$ is dense (using that $(\mathbb{P}^2)^3$ is irreducible, which we haven't yet discussed).

Lemma 13.6. *The locus of collinear triples of points is a closed subvariety of $(\mathbb{P}^2)^3$.*

Proof. Let's now construct the closed subset corresponding to such triples. Explicitly, the locus of collinear triples is simply

$$\mathcal{Y} := \left\{ [x_0, x_1, x_2], [y_0, y_1, y_2], [z_0, z_1, z_2] \subset (\mathbb{P}^2)^3 : \det \begin{pmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{pmatrix} = 0 \right\}.$$

Since the determinant is a trihomogeneous polynomial, $\mathcal{Y}$ defines a subvariety of $(\mathbb{P}^2)^3$. □

So, the locus of triples of collinear points is indeed a proper subvariety of $(\mathbb{P}^2)^3$, implying the complement (i.e., the locus of non-collinear triples) is open, and so a general member of this family does not correspond to a triple of collinear points. □

Example 13.7. In this example, we will explain why a general plane conic has rank 3.

Recall a plane conic is the zero locus in $\mathbb{P}^2$ of a single homogeneous quadratic polynomial on $\mathbb{P}^2$. We can express plane conics as $\mathbb{P}^5_{[a,b,c,d,e,f]}$. We have a universal family, as we saw last time

$$V\left(ax^2+by^2+cz^2+dxy+exz+fyz\right)\subset \mathbb{P}^5_{[a,b,c,d,e,f]}\times\mathbb{P}^2_{[x,y,z]}.$$

There are three kinds of plane conics. All are projectively equivalent to one of

(1) $V(x^2+y^2+z^2)$
(2) $V(x^2+y^2)$
(3) $V(x^2)$.

The first is a "smooth conic" the second is a union of two lines, and the third is a "double line".

Lemma 13.8. *A general plane conic has rank 3.*

Proof. It suffices to show that the locus of conics which has rank 1 or 2 is a closed subset.

Recall that a conic is associated to a symmetric bilinear form. In other words, we can think of $\mathbb{P}^5$ as the projectivization of the space of symmetric 3×3 matrices

$$\begin{pmatrix} a & d & e\\ d & b & f\\ e & f & c.\end{pmatrix}$$

Then, the locus where the determinant of this matrix is 0 precisely corresponds to rank 1 or 2 conics, and hence it is a closed subset of $\mathbb{P}^5_{[a,b,c,d,e,f]}$. □

Let's see one more example, which will be on the homework.

Example 13.9. Take $n>1$ and $d>0$. Then, a general polynomial $F(x_0,\ldots,x_n)$ of homogeneous degree d polynomial is irreducible (meaning it doesn't factor).

To show this, we want to show the locus of reducible polynomials is closed. Recall we have a universal family of hypersurface corresponding to degree d polynomials in $n+1$ variables. This forms a space $\mathbb{P}^{N_d}$ with $N_d=\binom{d+n}{n}-1$. For all a,b with $a+b=d$, $a,b\geq 1$, we get a map

$$\begin{aligned}\phi_{a,b}:\mathbb{P}^{N_a}&\to\mathbb{P}^{N_b}\\ (F,G)&\mapsto F\cdot G.\end{aligned}$$

Exercise 13.10. Show $\phi_{a,b}$ is a regular map.

Exercise 13.11. Show the locus of reducible polynomials is precisely the union of $\phi_{a,b}$ for all $a+b=d$ with $a,b\geq 1$.

Exercise 13.12. Show that the union of the images of $\phi_{a,b}$ with $a,b\geq 1$ and $a+b=d$ is not all of $\mathbb{P}^{N_d}$.

Very often, you might hear people say, "a given collection of polynomials cut out a variety."

Question 13.13. Let $f_1,\ldots,f_k \in \Bbbk[x_1,\ldots,x_n]$. Let $X\subset \mathbb{P}^n$. Then what does the phrase "the polynomials $f_1,\ldots,f_k$ cut out X"?

13.3. **Introduction to the Nullstellensatz.** There are two possible meanings for this phrase:

(1) The variety X is the common zero locus of $f_1,\ldots,f_k$.
(2) The polynomials $f_1,\ldots,f_k$ generate the ideal $I(X)$ of all polynomials vanishing on X.

We want to relate these two interpretations. Here is an alternate way to phrase this distinction.

Remark 13.14 (Important Remark)**.** We have a correspondence

$$\{\text{ closed subvarieties of } X\subset \mathbb{A}^n\} \overset{V}{\underset{I}{\updownarrow}} \{\text{ ideals } I\subset \Bbbk[x_1,\ldots,x_n]\} \tag{13.3}$$

This correspondence is not a bijection. By definition, $V\circ I=\mathrm{id}$. But, $I\circ V\neq \mathrm{id}$. As an example, the ideal x^2 satisfies $(x)=I\circ V(x^2)$, which is a different ideal.

We are almost ready to state the Nullstellensatz, but we need the notion of the radical of an ideal.

Definition 13.15. Suppose $I\subset R$ is any ideal, we define the **radical of** I to be

$$\sqrt{I} := \{x\in R : \text{ there exists some } n\in\mathbb{Z} \text{ with } x^n\in I\}.$$

Theorem 13.16 (The Nullstellensatz)**.** *Start with an ideal* $J\subset \Bbbk[x_1,\ldots,x_n]$. *Consider the associated ideal*

$$I(V(J)) = \{g\in \Bbbk[x_1,\ldots,x_n] : \textit{ if } f(x)=0 \textit{ for all } f\in J \textit{ then } g(x)=0\}.$$

Then $I(V(J))=\sqrt{J}$.

Exercise 13.17. We have a containment $\sqrt{J} \subset I(V(J))$.

The hard part of the proof of the Nullstellensatz is to show $I(V(J)) \subset \sqrt{J}$. We will see a proof next time.

Example 13.18. Among homogeneous quadratic polynomials, a polynomial is determined by its zero locus, up to scalars.

14. 2/22/16

14.1. **Review and Overview of Coming Attractions.** Today, we'll set up more framework for the commutative algebra we'll need. On Wednesday, we'll give the proofs.

Now, recall we have a two way correspondence (which is not a bijection!)

$$\begin{array}{c} \{\text{ varieties } X \subset \mathbb{A}^n\} \\ V \updownarrow I \\ \{\text{ ideals } I \subset \Bbbk[x_1, \ldots, x_n]\} \end{array} \tag{14.1}$$

Recall the failure of this to be a bijection is given by nilpotents (elements whose power are 0. We say an ideal is **radical** if $\sqrt{I} = I$, where

$$r(I) = \{f : f^m \in I \text{ for some } m \in \mathbb{Z}\}.$$

More precisely, this is given by the Nullstellensatz:

Theorem 14.1. *For $J \subset \Bbbk[x_1, \ldots, x_n]$, we have*

$$I(V(J)) = \sqrt{J}.$$

Remark 14.2. There are two ways to "make this into a bijection." On the one hand, by the Nullstellensatz, we have a bijection

$$\begin{array}{c} \{\text{ varieties } X \subset \mathbb{A}^n\} \\ \downarrow \\ \{\text{ radical ideals } I \subset \Bbbk[x_1, \ldots, x_n]\} \end{array} \tag{14.2}$$

The classical solution is to just restrict varieties to the vanishing loci of radical ideals.

However, in the 1960's, Grothendieck came along with a brilliant idea of enlarging the class of varieties. That is, he created a new object called a scheme and showed there was a bijection

$$\begin{array}{c} \left\{ \text{schemes } X \subset \mathbb{A}^n_{\Bbbk} \right\} \\ \big\downarrow \\ \{ \text{ideals } I \subset \Bbbk[x_1, \ldots, x_n] \} \end{array} \tag{14.3}$$

But, Grothendieck took this further! He got rid of both the restriction that I be an ideal in $\Bbbk[x_1, \ldots, x_n]$ and replaces this by an arbitrary ring.

$$\begin{array}{c} \{ \text{affine schemes} \} \\ \big\downarrow \\ \{ \text{commutative rings} \} \end{array} \tag{14.4}$$

To see more about this, check out Vakil's The Rising Sea or Eisenbud-Harris Geometry of Schemes, chapters I and II. (That is, we remove the hypotheses of nilpotent free, finitely generated, over a field.)

14.2. **What scissors are good for: Cutting out.**

Definition 14.3. We say a collection of polynomials

$$f_1, \ldots, f_k \in \Bbbk[x_1, \ldots, x_n]$$

cut out $X \subset \mathbb{A}^n$ **set theoretically** if $V(f_1, \ldots, f_n) = X$.

Definition 14.4. We say a collection of polynomials

$$f_1, \ldots, f_k \in \Bbbk[x_1, \ldots, x_n]$$

cut out $X \subset \mathbb{A}^n$ **scheme theoretically** if $(f_1, \ldots, f_k) = I(X)$.

Remark 14.5. Don't worry about the words "scheme theoretically." You don't have to know what a scheme is. This is just terminology.

Next, we'll discuss what it means for polynomials to cut out a variety in projective space.

Let $S = \Bbbk[x_0, \ldots, x_n]$. Inside S, define the subset S_m to be the homogeneous polynomials of degree m. We have $S = \oplus_{m \geq 0} S_m$.

Definition 14.6. We say $I \subset S$ is a **homogeneous ideal** if I is generated by homogeneous polynomials.

Exercise 14.7. Show that $I \subset S$ is a homogeneous ideal if and only if $I = \oplus_{m \geq 0} I_m$ where $I_m = I \cap S_m$.

Definition 14.8. For $S = \Bbbk[x_0, \ldots, x_n]$, we define the irrelevant ideal as $(x_0, \ldots, x_n)$. It is called the irrelevant ideal because $V(x_0, \ldots, x_n) = \emptyset$.

We have a bijective correspondence

$$\begin{array}{c} \{\text{ projective varieties } X \subset \mathbb{P}^n\} \\ V \updownarrow I \\ \{\text{ homogeneous ideals } I \subset \Bbbk\,[x_0, \ldots, x_n]]\}. \end{array} \tag{14.5}$$

There are two obstructions to this being a bijection.

(1) First, an ideal will have the same vanishing locus as its radical, so V is not injective.
(2) Two ideals with the same saturation (to be defined soon) will have the same vanishing locus. As an example, if you take an ideal and through away the first finitely many graded pieces of this ideal, these two ideals will have the same vanishing locus. That is, I and $I'_m := \oplus_{m \geq m_0} I_m$ have the same vanishing locus.

Definition 14.9. For $I \subset S = \Bbbk[x_0, \ldots, x_n]$ a homogeneous ideal, the **saturation of** I is

$$\{f \in \Bbbk[x_0, \ldots, x_n] : \text{ there exists some degree } d \text{ so that } fg \in I \text{ for all } g \in S \text{ of degree at least } d\}$$

We define this set to be $\mathrm{Sat}(I) \subset \Bbbk[x_0, \ldots, x_n]$.

Exercise 14.10. Let $I \subset S$ be a homogeneous ideal. Then $\mathrm{Sat}(I) \subset S$ is a homogeneous ideal.

We now make a definition of "cutting out" for projective variety.

Definition 14.11. Let $f_1, \ldots, f_k \in S$ be homogeneous polynomials. Then, $f_1, \ldots, f_k$

(1) **cut out** $X \subset \mathbb{P}^n$ **set theoretically** if $X = V(f_1, \ldots, f_k)$.
(2) **cut out** $X \subset \mathbb{P}^n$ **scheme theoretically** if $\mathrm{Sat}(f_1, \ldots, f_k) = I(X)$.
(3) **generate the homogeneous ideal of** $X \subset \mathbb{P}^n$ if $f_1, \ldots, f_k = I(X)$.

Exercise 14.12. (1) Show that if $f_1, \ldots, f_k$ generate the homogeneous ideal of X then they cut out X scheme theoretically.

(2) Show that if $f_1, \dots, f_k$ cut out X scheme theoretically, they cut out X set theoretically.

Remark 14.13. The distinction between the projective and affine cases is that once you dehomogenize the polynomials in the projective case, you get the affine case. (This uses the notion of localization.)

14.3. Irreducibility.

Example 14.14 (Irreducibility of Hypersurfaces)**.** Start with the case of a hypersurface $X = V(f)$ with $f \in \mathbb{k}[x_1, \dots, x_n]$. That is, we can write $f = \prod_i f_i^{m_i}$ where f_i are irreducible. Then, we can write $X = V(f) = \cup_i V(f_i)$.

Question 14.15. Is there an analogous way to break an arbitrary variety to pieces corresponding to irreducible parts?

To answer this, we'll give a geometric characterization of this property.

Definition 14.16. Let X be a topological space. Then, X is **irreducible** if one cannot write X as the union of two proper closed subsets, meaning both subsets are neither empty nor all of X. If a topological space is **reducible** if it is not irreducible.

Example 14.17. If $f, g \in S$ we have $X = V(fg) = V(f) \cup V(g)$ is reducible if neither f nor g is a factor of the other.

Exercise 14.18. Let $X \subset \mathbb{A}^n$ be irreducible. Then, $f \notin I(X)$ implies $V(f, I(X)) \subset X$ is a proper closed subset.

Exercise 14.19. Show that X is irreducible if and only if $I(X)$ is prime. *Hint:* Show that for any two $f, g \notin I(X)$ we have $fg \notin I(X)$, using the definition of irreducibility.

Theorem 14.20. *Any variety $X \subset \mathbb{A}^n$ is uniquely expressible as a union of a finite collection of irreducible varieties. Equivalently, any radical ideal $I \subset \mathbb{k}[x_1, \dots, x_n]$ is uniquely expressible as a n intersection of prime ideals.*

Proof. Deferred to next time. □

Exercise 14.21. Show Theorem 14.20 holds for the case of hypersurfaces $X \subset \mathbb{A}^n$. *Hint:* Use unique factorization.

Remark 14.22. Why is a ring called a ring? Why is a field called a field? Why is an ideal called an ideal?

The answer is usually buried in some obscure history and lost in translation. But, ideals are called ideals largely due to Theorem 14.20. For rings in number fields, and other examples, unique factorization fails. However, in rings of integers of number fields, (or more generally for Dedekind domains,) one can factor ideals uniquely. People originally called ideals "ideal numbers."

15. 2/24/16

15.1. **Overview.** Today, we're bringing in the commutative algebra. There are three parts of the plan for today.

(1) Prime decomposition of radical ideals
(2) The Nullstellensatz
  (a) weak Nullstellensatz
  (b) Nullstellensatz

15.2. **Preliminaries.** We'll start by discussing the notion of a Noetherian rings. The category of all rings is a bit too big for us. We prefer to work with Noetherian rings, which is a certain sort of finiteness condition on the ring. Nevertheless, most of the rings we work with will be Noetherian. For example, $\mathbb{k}[x_1, \ldots, x_n]/I$ is Noetherian.

Definition 15.1. A commutative ring R is **Noetherian** if every increasing sequence of ideals stabilizes. That is, if we have a sequence of ideals

$$I_1 \subset I_2 \subset \cdots$$

then there is some finite k so that $I_k = I_j$ for all $j > k$.

Remark 15.2. In the case $R = \mathbb{k}[x_1, \ldots, x_n]$, we have a correspondence between ideals and varieties, and once we know it is Noetherian, we will know any decreasing sequence of subvarieties of $\mathbb{A}^n$ stabilizes. That is, if we have

$$\mathbb{A}^n \supset X_1 \supset X_2 \supset X_3 \supset \cdots$$

then there exists k so that $X_j = X_k$ for all $j > k$.

More generally, we say a topological space is Noetherian if every decreasing sequence of closed subsets terminates.

Warning 15.3. Most "reasonable" topological spaces will not be Noetherian. For example, any manifold of dimension at least 1 will not be Noetherian.

We start with some equivalent properties

Lemma 15.4. *Let R be a commutative ring. The following are equivalent.*

(1) R is Noetherian
(2) Every ideal $I \subset R$ is finitely generated

(3) *For any finitely generated* R *module* M *and submodule* $N \subset M$, *we have that* N *is finitely generated.*
(4) *Any collection of ideals of* R *has a maximal element.*

Proof. For $(1) \implies (2)$, if the ideal were not finitely generated, we could get an infinite sequence of strictly increasing ideals, each not equal to the next.

Exercise 15.5 (Tricky Exercise)**.** Show the remaining implications. See, for example, Gaitsgory's Math 123 notes from 2013, or most standard commutative algebra textbooks, like Atiyah MacDonald.

□

Theorem 15.6 (Hilbert Basis Theorem)**.** *If* R *is Noetherian then* $R[x]$ *is Noetherian.*

Proof. This takes some work. See, for example, Gaitsgory's 123 notes. □

Lemma 15.7. *If* R *is Noetherian, then* R/I *is Noetherian.*

Proof.

Exercise 15.8. Prove this. **Hint: Any sequence of ideals in** R/I **lifts to a sequence in** R**, and use the Noetherian condition on** R**.**

□

Corollary 15.9. *Any rings of the form* $\Bbbk[x_1, \ldots, x_n]/I$ *is Noetherian.*

Proof. Combine the proceeding Lemma and Theorem. □

15.3. **Primary Decomposition.**

Proposition 15.10. *Let* $I \subset \Bbbk[x_1, \ldots, x_n]$ *be a radical ideal. Then, we can express* I *as* $I = \cap_{i=1}^m \mathfrak{p}_i$ *with* $\mathfrak{p}_i$ *prime.*

Remark 15.11. Further, this expression is unique if we assume the $\mathfrak{p}_i$ are minimal among primes containing I, although we won't prove this.

Remark 15.12. The above statement holds more generally in an arbitrary Noetherian ring.

Further, we can generalize this proposition to arbitrary ideals, but for that we need a notion of primary ideals, which is discussed in Atiyah MacDonald or Eisenbud's commutative algebra with a view toward algebraic geometry.

Proof of Proposition 15.10. Consider the

$$\{ \text{ radical ideals } I \subset \Bbbk[x_1, \ldots, x_n] : \text{ I is not expressible as a finite intersection of prime ideals } \}$$

Let I_0 be the maximal element of this collection. If I_0 is not prime. That is, if there exist $a, b \in \Bbbk[x_1, \ldots, x_n]$ so that $a, b \notin I_0$ and $ab \in I_0$. Set

$$I_1 := \sqrt{(I_0, a)} \supsetneq I_0$$
$$I_2 := \sqrt{(I_0, b)} \supsetneq I_0$$

Now, $I_1 = \cap_i \mathfrak{p}_i, I_2 = \cap_j \mathfrak{q}_j$. It suffices to show that $I_1 \cap I_2 = I_0$, since then I_0 will be the intersection of the primes $\mathfrak{p}_i$ and $\mathfrak{q}_i$. This is the following lemma.

Lemma 15.13. *With I_1, I_2, I_0 as defined in the proof above $I_1 \cap I_2 = I_0$.*

Proof. Say $f \in I_1 \cap I_2$. Then, because I_1 is the radical of (I_0, a) and I_2 is the radical of (I_0, b) there exists m, n with

$$f^m = g_1 + h_1 a$$
$$f^n = g_2 + h_2 b.$$

with $g_1, g_2 \in I_0$. Therefore, $f^{m+n} = g_1 g_2 + g_1 h_1 a + g_2 h_1 a + h_1 h_2 ab \in I_0$. Therefore, because I_0 is radical, $f^{m+n} \in I_0 \implies f \in I_0$. □

□

Remark 15.14. Geometrically, this is saying a variety is a union of finitely many irreducible components which correspond to the minimal primes containing the ideal.

15.4. **The Nullstellensatz.** Recall the main theorem:

Theorem 15.15 (The Nullstellensatz)**.** *Let $J \subset \Bbbk[x_1, \ldots, x_n]$ be any ideal. Then,*

$$I(V(J)) = \sqrt{J}.$$

It's clear that $\sqrt{J} \subset I(V(J))$. We prove this by reducing it the the weak Nullstellensatz.

Theorem 15.16 (Weak Nullstellensatz)**.** *If $J \subset \Bbbk[x_1, \ldots, x_n]$ satisfies $V(J) = \emptyset$ then $\sqrt{J} = (1)$*

Proof of Theorem 15.15, assuming Theorem 15.16. We use a trick called the trick of Rabinowitsch. Let $I \subset \Bbbk[x_1, \ldots, x_n]$ be any ideal. We claim that if $f \in \Bbbk[x_1, \ldots, x_n]$ vanishes on $V(I)$ then $f^m \in I$ for some m.

To show this claim, we introduce an auxiliary subvariety. Consider

$$\Sigma = \left\{(x_1, \ldots, x_n, y) \in \mathbb{A}^{n+1} = \mathbb{A}^n_x \times \mathbb{A}^1_y : yf(x_1, \ldots, x_n) = 1.\right\}$$

we have a projection

$$\begin{array}{c} \Sigma \\ \downarrow \\ \mathbb{A}^n \end{array} \tag{15.1}$$

This image is the complement of the zero locus of f.

So, since $f = 0$ on $V(I)$, we obtain that

$$V(I, y \cdot f(x_1, \ldots, x_n) - 1) = \emptyset \subset \mathbb{A}^{n+1}.$$

By the weak Nullstellensatz, we have $1 \in (I, y \cdot f(x) - 1)$. So, $1 = g_0(yf(x) - 1) + \sum_i h_i y^i$ with $g_0 \in \mathbb{k}[x_1, \ldots, x_n, y], h_i \in I$. Here, the term $\sum_i h_i y^i$ is just grouping terms of our summand in I by power of y.

Now, substituting $1/f$ for y, or more precisely working in the ring $\mathbb{k}[x_1, \ldots, x_n, y]/(yf - 1) \cong \mathbb{k}[x_1, \ldots, x_n]$, we have

$$1 = \sum_{i=1}^{m} h_i f^{-i}$$

and multiplying by f^m, we have $f^m \in I$ since

$$f^m = \sum_{i=0}^{m} h_i f^{m-i}$$

and the right hand side lies in I. □

To conclude the proof of the Nullstellensatz, we only need prove the weak Nullstellensatz.

Proof of Theorem 15.16. We want to show that if $I \subset \mathbb{k}[x_1, \ldots, x_n]$ and $V(I) = \emptyset$ implies $I = (1)$. We reduce this to the following alternate form of the Nullstellensatz given in Proposition 15.17. Since every ideal $I \subsetneq (1)$ is contained in some maximal ideal, we have $I \subsetneq (1)$ implies that $V(I) \neq \emptyset$. □

Proposition 15.17. *Any maximal ideal* $\mathfrak{m} \subset \mathbb{k}[x_1, \ldots, x_m]$ *is of the form* $(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$.

Proof. We'll probably see this proof next time. □

Remark 15.18 (Important remark)**.** The statements about the Nullstellensatz crucially depends on the field $\mathbb{k}$ being algebraically closed.

For example, if we take (x^2+1) over $\mathbb{A}^1_{\mathbb{R}}$ this ideal is in fact maximal in $\mathbb{R}[x]$, but it is not of the form $x - a$.

16. 2/26/15

16.1. **Review of the Nullstellensatz.** Before getting back to the proof of the Nullstellensatz, we mention the following application of the Nullstellensatz.

Corollary 16.1. *Suppose* $X \subset \mathbb{A}^n$ *is an affine variety. Let* $I \subset \mathbb{k}[x_1, \ldots, x_n]$ *and* $A(X) = \mathbb{k}[x_1, \ldots, x_n]/I(X)$*. Then, the ring of regular functions on* X *is* $A(X)$*.*

Proof. Certainly all functions in $A(X)$ are (distinct) regular functions on X. The completion of the proof is given in the course textbook, "A First Course." □

Now, we return to finishing up the proof of the Nullstellensatz. Recall that last time we reduced the proof to the following assertion.

Proposition 16.2. *Let* $\mathbb{k}$ *be an algebraically closed field. Then, the only maximal ideals of*

$$\mathbb{k}[x_1, \ldots, x_n]$$

are of the form

$$(x_1 - a_1, \ldots, x_n - a_n)$$

for $a_i \in \mathbb{k}$*.*

16.2. **Preliminaries to Completing the proof of the Nullstellensatz.** Before getting to the proof of this, we recall a few things from commutative algebra.

For the moment, we are no longer assuming our fields are algebraically closed or of characteristic 0.

Definition 16.3. Suppose we have a field extension $K \subset L$. We say L is **algebraic** over K if every element of L satisfies a polynomial over K. If L is not algebraic over K, we say L is **transcendental**.

Remark 16.4. If L/K is transcendental then L is not a finitely generated K algebra. For example, $L = K(x)$ is transcendental, and as a not too difficult fact, all transcendental field extensions are algebraic extensions of $K(x_i)$.

Lemma 16.5. $K(x)$ *is not finitely generated as a* K *algebra.*

Proof. Say $z_1, \ldots, z_k \in K(x)$ generate $K(x)$ as a K algebra. Write

$$z_i = \frac{p_i(x)}{q_i(x)}.$$

Then, given any irreducible polynomial $f(x) \in K[x]$, we can write $\frac{1}{f(x)}$ as a polynomial in the z_i's. Clearing denominators, we see f divides at least one of the denominators q_i, since f is a factor of the right hand side, while the left hand side is a product of $q_i(x)$.

But then, there are infinitely many irreducible polynomials, a contradiction.

To see there are infinitely many irreducible polynomials, if K is infinite, just take those of the form $x - a$ for $a \in K$. If K is finite, it is p^n for some prime p, and then we can take the cyclotomic polynomials which are the highest degree irreducible factors of polynomials of the form $(x^{p^n})^m - x$. □

16.3. **Proof of Nullstellensatz.** We now return to the proof of the Nullstellensatz.

Proof of Proposition 16.2. In order to complete the proof, we'll need the following Lemma.

Remark 16.6. Here's a nice quote from Joe: The following lemma is

(1) True
(2) Implies the Nullstellensatz

which are the two qualities a good lemma should have! That said, it's completely opaque.

Lemma 16.7. *If R is a Noetherian ring and S is any subring $R \subset S \subset R[x_1, \ldots, x_n]$. Then, if $R[x_1, \ldots, x_n]$ is finitely generated as an S module, then S is finitely generated as an R algebra.*

Proof. Say $y_1, \ldots, y_k \in R[x_1, \ldots, x_n]$ generate $R[x_1, \ldots, x_n]$ as an S module. Then, we can write $x_i = \sum_j a_{ij} y_j$ with $a_{ij} \in S$ and we can write $x_i x_j = \sum_k b_{ijk} y_k$ with $b_{ijk} \in S$. Let $S_0 \subset S$ be the subalgebra generated over R by a_{ij}, b_{ijk}. That is, we have

$$R \subset S_0 \subset S \subset R[x_1, \ldots, x_n].$$

We know that S_0 is a finitely generated R algebra. This implies S_0 is Noetherian (although we did not yet know S is Noetherian!). Note that $R[x_1, \ldots, x_n]$ is still finitely generated as a module over S_0 by the elements $y_1, \ldots, y_k$. By the Noetherian condition, we have that S is a finitely generated S_0 module. Hence, S is finitely generated as an R algebra, generated by S_0 together with $y_1, \ldots, y_k$. □

We now conclude the proof of Proposition 16.2. Now, let $\Bbbk$ be an algebraically closed field. Let $\mathfrak{m} \subset \Bbbk[x_1, \ldots, x_n]$. We want so show

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$$

for $a_i \in \Bbbk$. The above condition is equivalent to saying that $\Bbbk[x]/\mathfrak{m} \cong \Bbbk$, since every such maximal ideal of the above form, and all such ideals have quotient which is $\Bbbk$.

First, if $\Bbbk[x]/\mathfrak{m}$ is algebraic over $\Bbbk$, then this is algebraic, and hence isomorphic to $\Bbbk$ because $\Bbbk$ is algebraically closed.

Otherwise, $\Bbbk[x]/\mathfrak{m}$ is transcendental over $\Bbbk$ because $\Bbbk[x]/\mathfrak{m}$ is not finitely generated as a K algebra, by applying Lemma 16.7 to the inclusion $K \subset L$. □

16.4. **Grassmannians.** Let V be a vector space of dimension n over $\Bbbk$. Recall that we have introduced $\mathbb{P}V$, which, as a set, is all 1 dimensional subspaces of V.

Definition 16.8. We define the Grassmannian

$$G(k, V) := \{k \text{ dimensional subspaces of } V\}.$$

Remark 16.9. Note the spelling of Grassmannian. Two s's and two n's!

Example 16.10. If $k = 1$, we have $G(1, V) = \mathbb{P}V$. If $k = n - 1$, we have $G(1, V) = \mathbb{P}V^\vee$.

Our first order of business is to give $G(k, V)$ the structure of a variety.

Lemma 16.11. *Let V have dimension n over $\Bbbk$. The grassmannian $G(k, V)$ is a projective subvariety of $\mathbb{P}^{\binom{n}{k}-1}$.*

In order to prove Lemma 16.11, we will need some preliminaries on multilinear algebra. In the following, we do not require $\Bbbk$ is algebraically closed, although at one point, which we point out, we will assume characteristic 0.

Definition 16.12. Let V, W be vector spaces over $\Bbbk$. We have the following three equivalent characterizations of $V \otimes W$.

(1) Choose bases $v_1, \ldots, v_m$ for V and $w_1, \ldots, w_n$ for W. Then, define the tensor product to be the $\Bbbk$ vector space with $\Bbbk$ basis $\{v_i \otimes W_j\}_{i,j}$.
(2) Define

$$V \otimes W := \Bbbk\langle\{v \otimes w : v \in V, w \in W\} / \sim\rangle$$

where ~ is the equivalence relation generated by

$$\begin{aligned}(\lambda v) \otimes w - \lambda(v \otimes w)\\ v \otimes (\lambda w) - \lambda(v \otimes w)\\ (v_1 + v_2) \otimes w - (v_1 \otimes w + v_2 \otimes w)\\ v \otimes (w_1 + w_2) - (v \otimes w_1 + v \otimes w_2).\end{aligned}$$

(3) (This definition is given via a universal property). We define $V \otimes W$ to be the unique vector space, up to unique isomorphism for any bilinear map $V \times W \to U$, with U a vector space,

$$\begin{array}{ccc} V \times W & \longrightarrow & U \\ & \searrow \qquad \nearrow & \\ & V \otimes W & \end{array} \tag{16.1}$$

so that for any bilinear map $V \times W \to U$ there is a unique linear map $V \otimes W \to U$.

Exercise 16.13 (Tricky exercise, if you haven't seen it before)**.** Show the three characterizations of the tensor product given in Definition 16.12 are equivalent.

Proof Lemma 16.11. First, we give an injective map $G(k, V) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$. We'll finish the proof next time. □

17. 2/29/16

17.1. **Overview and a homework problem.** Today and Wednesday, we'll cover 1. On Friday, we'll move on to rational maps. Next week, we'll cover chapter 7 and possibly chapter 8. After break, we'll start on part 2, which begins with chapter 11, talking about things like the dimension of a variety, Hilbert polynomials, smoothness On a first pass, we'll skip chapters 9 and 10.

But, before continuing on Grassmannians, we'll go over a homework problem.

Question 17.1 (Homework 4, Question 5)**.** Let $d \geq 1$ and $m \leq \binom{d+2}{2}$. A general collection of points $p_1, \ldots, p_m \in \mathbb{P}^2$ imposes independent conditions on the vector space of degree d polynomials.

17.1.1. *Solution.* Let us start by saying what it means for these points to **impose independent conditions**. We have a map

$$\left\{ \text{homogeneous polynomials of degree } d \text{ on } \mathbb{P}^2 \right\} \xrightarrow{\phi} k^m$$

given by evaluation at each of the m points. Imposing independent conditions mean that the above map is surjective. In other words, the kernel of the map has codimension m. This is useful because it tells us we can interpolate degree d polynomials through points. First, we introduce the parameter space of m tuples of points, which is $(\mathbb{P}^2)^m$.

To complete the problem, there are three parts to check. Let $U = \left\{(p_1, \ldots, p_m) : p_i \text{ impose independent conditions } \subset (\mathbb{P}^2)^m\right\}$.

(1) Show that U is open.
(2) Show $U \neq \emptyset$.
(3) Show that U is dense. If the above two results are satisfied, this is automatically satisfied if the parameter space $(\mathbb{P}^2)^m$ is irreducible.

We now verify these three claims.

(1) Write out a matrix representative of ϕ. That is choose a basis $f_0, \ldots, f_N$ of polynomials of degree d, with $N = \binom{d+2}{2} - 1$. So, we have a matrix of the form

$$\begin{pmatrix} f_0(p_1) & \cdots & f_N(p_m) \\ \vdots & \ddots & \vdots \\ f_0(p_1) & \cdots & f_n(p_m) \end{pmatrix}$$

To see this map is not surjective, this is equivalent to saying this matrix is rank deficient, which means all determinants of $m \times m$ submatrices vanish. Now, the determinants are simply multihomogeneous polynomials of the m coordinates of the points. In other words, each maximal minor is a multihomogeneous polynomial on $(\mathbb{P}^2)^m$, and so $(\mathbb{P}^2)^m \setminus U$ is the vanishing locus of these polynomials. In other words, this is a closed subset of $(\mathbb{P}^2)^m$.

(2) Here we'll show such a configuration exists, without writing one down. We induct on m. If $m = 1$, this is automatic, since there is no point in $\mathbb{P}^2$ where every polynomial vanishes. This is the base case. Now, assuming we've chosen $m - 1$ points imposing independent conditions. To choose the mth point, we just need to choose it to not be in the common zero locus of the polynomials through the first $m - 1$ points. Such a point

because the common zero locus is a proper subvariety, since any single degree d curve is a proper subvariety. Choosing the mth point outside of that zero locus gives m points imposing independent conditions.

(3) This holds because $\mathbb{P}^2$ is irreducible, and products of irreducible varieties are irreducible.

17.2. **Multilinear algebra, in the service of Grassmannians.** Recall, the equivalent definitions of tensor products as discussed last time. Let V, W be two vector spaces of dimension n and m. The tensor product is another vector space $V \otimes W$ with a bilinear map

$$V \times W \to V \otimes W$$

which can be characterized in any of the three following ways.

(1) If $V = \langle e_1, \ldots, e_n \rangle, W = \langle f_1, \ldots, f_m \rangle$, and $V \otimes W = \langle e_i \otimes f_j \rangle_{i,j}$. The bilinear map is

$$\begin{aligned} V \times W &\to V \otimes W \\ \left(\sum c_i e_i, d_j f_j \right) &\mapsto \sum_{i,j} c_i d_j e_i \otimes f_j \end{aligned}$$

(2) Define

$$V \otimes W := \mathbb{k}\langle \{ v \otimes w : v \in V, w \in W \} / \sim \rangle$$

where $\sim$ is the equivalence relation generated by

$$\begin{aligned} (\lambda v) \otimes w - \lambda(v \otimes w) \\ v \otimes (\lambda w) - \lambda(v \otimes w) \\ (v_1 + v_2) \otimes w - (v_1 \otimes w + v_2 \otimes w) \\ v \otimes (w_1 + w_2) - (v \otimes w_1 + v \otimes w_2). \end{aligned}$$

(3) (This definition is given via a universal property). We define $V \otimes W$ to be the unique vector space, up to unique isomorphism for any bilinear map $V \times W \to U$, with U a vector space,

$$\begin{array}{ccc} V \times W & \longrightarrow & U \\ & \searrow \quad \nearrow & \\ & V \otimes W & \end{array} \tag{17.1}$$

so that for any bilinear map $V \times W \to U$ there is a unique linear map $V \otimes W \to U$.

Now, we'll come to symmetric and antisymmetric powers of a vector space. As a special case of the tensor product, for V a vector spaces we can form $V^{\otimes 2} := V \otimes V$. In general, we define $V^{\otimes m} := V \otimes \cdots \otimes V$, with m copies of V.

Now, inside $V \otimes V$, we have symmetric tensors, which is the span of expressions of the form $v \otimes w + w \otimes v$. This is the **second symmetric power** denoted $\operatorname{Sym}^2 V$. We have an isomorphism

$$\operatorname{Sym}^2 V \cong V \otimes V / \langle v \otimes w - w \otimes v \rangle.$$

(Note that this assumes the characteristic of the field is not 2).

We also have skew-symmetric tensors which are generated by those of the form $\langle v \otimes w - w \otimes v \rangle$. This is the second wedge power, denoted $\wedge^2 V$. We can write

$$\wedge^2 V = V \otimes V / \langle v \otimes w + w \otimes v \rangle.$$

We now generalize this to all $k \geq 0$.

Definition 17.2. The k**th symmetric power** of a vector space V is the subspace of symmetric tensors in $V^{\otimes k}$. We define the k**th alternating power** to be the anti-symmetric tensors in $V^{\otimes k}$, that is, tensors which change sign when you transpose two components of V.

We notate a wedge power of k vectors $v_1, \ldots, v_k$ as $v_1 \wedge \cdots \wedge v_k \in \wedge^k V$, which is the image of $v_1 \otimes \cdots \otimes v_k$ under the map $V^{\otimes k} \to V^{\otimes k} / \langle \sum_{\sigma \in S_k} (-1)^{sgn(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \rangle$. We write the analogous symmetric product of vectors as $v_1 \cdots v_k$ (with no symbols between them).

Remark 17.3. In the case $k = 2$, we have $V^{\otimes 2} \cong \operatorname{Sym}^2 V \oplus \wedge^2 V$. However, for $k > 2$, $V^{\otimes k} \not\cong \operatorname{Sym}^k V \oplus \wedge^k V$. In fact, the dimension of the left side will be more than that of the right hand side when $\dim V > 1$.

Remark 17.4. For V n dimensional, we have

$$\operatorname{Sym}^2 V^\vee = \{ \text{ symmetric bilinear forms on } V \}$$

$\wedge^2 V = \{$ alternating bilinear forms on $V\}$

Also, $\wedge^n V$ is 1 dimensional, and corresponds to the determinants of matrices.

We have bilinear maps

$$\begin{aligned} \wedge^k V \times \wedge^l V &\to \wedge^{k+l} V \\ (w, v) &\mapsto w \wedge v. \end{aligned}$$

17.3. **Grassmannians.** We can now describe Grassmannians, using our new multilinear algebra tools.

We want to define the grassmannian $G(k, V)$ as the k dimensional linear subspaces of V. We will construct a set theoretic inclusion $G(k, V) \hookrightarrow \mathbb{P}^N$ for some large N where the image is the zero locus of homogeneous polynomials on $\mathbb{P}^N$. To obtain this inclusion, we will need exterior products.

For $\Lambda \subset V$, a k dimensional subspace, we want to choose a basis $v_1, \ldots, v_k$ for Λ and construct the map

$$\begin{aligned} G(k, V) &\to \mathbb{P} \wedge^k V \\ [\Lambda] &\mapsto [v_1 \wedge \cdots v_k] . \end{aligned}$$

Note that if we chose a different basis for Λ, the image of Λ changes by the determinant of the change of basis map, so this is well defined up to scalar multiple. As a cleaner description, we have an inclusion $\Lambda \hookrightarrow V$, and so we get an associated functorial inclusion $\wedge^k V \hookrightarrow \wedge^k V$. On Wednesday, we'll see the image of this map is given by homogeneous polynomials.

18. 3/2/16

18.1. **Review.** Recall the following statements from multilinear algebra, discussed last class. Let V be a vector space and $V^{\otimes k} = V \otimes \cdots \otimes V$, with the right hand side having k copies of V. Note that S_k, the symmetric group on k elements, acts on $V^{\otimes k}$. We define

$$\begin{aligned} \operatorname{Sym}^k V &:= \left\{ \eta \in V^{\otimes k} : \sigma(\eta) = \eta, \text{ for all } \eta \in S_k \right\} \\ &\cong V^{\otimes k} / \langle \eta - \sigma(\eta) : \eta \in V^{\otimes k}, \sigma \in S_k \rangle. \end{aligned}$$

The latter isomorphism depends on the characteristic not being 0. The map is given by reduction, and the inverse map is given by summing over all preimages and dividing by $k!$. There are two reasons we don't care about the difference between these definitions

(1) We're working in characteristic 0
(2) we're going to be working modulo scalars anyway.

Similarly, recall

$$\wedge^k V = \left\{ \eta \in V^{\otimes k} : \sigma(\eta) = (-1)^\sigma \cdot \eta \text{ for all } \sigma \in S_k \right\}.$$

Observe the following three useful facts about wedge products.

(1) If $e_1, \ldots, e_k$ are a basis for V, then

$$\left\{ e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < \cdots < i_k \le n \right\}$$

forms a basis for $\wedge^k V$.

(2) We have a bilinear map

$$\wedge^k V \times \wedge^l V \to \wedge^{k+l} V$$

which is symmetric if kl is even and skew symmetric if kl is odd.

(3) Note that $v \wedge v = 0 \in \wedge^2 V$.

18.2. **The grassmannian is a projective variety.**

Definition 18.1. Notationally, if $\Lambda \subset \mathbb{P}V$ is a plane, we let $[\Lambda] \in G(k, V)$ denote the corresponding point. Very often, we will confuse the two, and simply notate Λ as a point of $G(k, V)$.

Now, we'll see why the grassmannian is a projective variety.

Definition 18.2. For V an n dimensional vector space, define

$$G(k, n) := G(k, V).$$

Lemma 18.3. *Let V be an n dimensional vector space. We have an inclusion of sets $G(k, V) \hookrightarrow \mathbb{P} \wedge^k V$. via either of the following two equivalent maps*

(1) Given $\Lambda = \langle v_1, \ldots, v_k \rangle \subset V$, define

$$\begin{aligned} \iota : G(k, V) &\to \mathbb{P} \wedge^k V \\ \Lambda &\mapsto [v_1 \wedge \cdots \wedge v_k] . \end{aligned}$$

(2) We define the map

$$\begin{aligned} \iota : G(k, V) &\to \mathbb{P} \wedge^k V \\ \Lambda &\mapsto \wedge^k \Lambda \end{aligned}$$

where we think of $\wedge^k \Lambda$ as a 1 dimensional subspace of $\wedge^k V$.

Proof. The first map defined is well defined because choosing a different basis changes the image by the scalar which is the determinant of the change of basis matrix.

Exercise 18.4. Verify that the two maps constructed agree. *Hint:* Choose a basis for Λ to relate the second to the first, and verify the first is independent of basis.

Exercise 18.5. Show this map is injective. *Hint:* Show two pure wedge products will be equal in $\mathbb{P} \wedge^k V$ only if they are related by a scalar.

□

Proposition 18.6. *The image of ι as defined in Lemma 18.3 is a projective variety.*

Proof. To show this, we will need a lemma.

Lemma 18.7. *Let $\eta \in \wedge^k V$ and $\nu \in V$. Then,*

$$\varepsilon = \nu \wedge \phi$$

for some $\phi \in \wedge^{k-1} V$ if and only if $\varepsilon \wedge \nu = 0$.

Proof. The forward direction is clear because $\nu \wedge \nu = 0$. To show the reverse direction, express express η in terms of a basis $e_1, \ldots, e_n$, with $\nu = e_1$. □

Given $\eta \in \wedge^k V$, consider the map

$$\begin{aligned} \phi_\eta : V &\to \wedge^{k+1} V \\ \nu &\mapsto \nu \wedge \eta. \end{aligned}$$

By Lemma 18.7, we have that η is totally decomposable if and only if $\dim \ker \phi_\eta = k$. Or, equivalently, $\operatorname{rk} \phi_\eta \leq n - k$.

Remark 18.8. This is actually saying something slightly stronger than the lemma. The lemma above says that η is divisible by each of k elements of a basis of this kernel, $\nu_1, \ldots, \nu_k$ separately, which means we can write it as $\eta = \nu_i \wedge w_i$ for each i. One then has to verify that in fact $\eta = c\nu_1 \wedge \cdots \wedge \nu_k$ for $c \in \Bbbk$ a scalar.

Exercise 18.9. Show that in fact $\eta = c\nu_1 \wedge \cdots \wedge \nu_k$ for $c \in \Bbbk$ a scalar.

So, we have that ϕ_η is a linear map depending linearly on η. In other words, we have a map

$$\begin{aligned} \phi : \wedge^k V &\to \mathrm{Hom}(V, \wedge^{k+1} V \\ \eta &\mapsto \phi_\eta. \end{aligned}$$

Then, the space of $n \times \binom{n}{k+1}$ matrices $G(k, V) \subset \mathbb{P} \wedge^k V$ is the zero locus of the pullback under ϕ of the $(n-k+1) \times (n-k+1)$ minors, where we are thinking of $\mathrm{Hom}(V, \wedge^{k+1} V)$ as matrices, and looking at the minors of these matrices. □

Remark 18.10. Henceforth, we will refer to the grassmannian as a projective variety using the map ι above, which realizes it as a projective variety.

In fact, the grassmannian is cut out by quadratic polynomials, although it takes a little more work to see this. These quadratic equations generating the homogeneous ideal of the grassmannian are called Plücker relations.

18.3. **Playing with grassmannian.** What makes the grassmannian $G(k, V)$ is so interesting to us is that it is a parameter space for linear spaces in $\mathbb{P}V$. To give this description, observe the following.

Remark 18.11. We have that

$$\begin{aligned} G(k, V) &= \{k \text{ dimensional subspaces } \Lambda \subset V\} \\ &= \{(k-1) \text{ planes in } \mathbb{P}V\}. \end{aligned}$$

When we want to think of the grassmannian as the latter set, we notate

$$\mathbb{G}(k-1, \mathbb{P}V) := G(k, V).$$

For V n dimensional, we notate

$$\mathbb{G}(k-1, n-1) := \mathbb{G}(k-1, \mathbb{P}V).$$

Example 18.12. We have $G(2,4) = \mathbb{G}(1,3)$ which is the space of lines in $\mathbb{P}^3$.

Remark 18.13. Recall that when we defined "parameter space" we mentioned that parameter space should be the base of a family, with the fibers the things that the points in the base parameterizes. Indeed, this happens with the grassmannian, and we'll now construct this as the universal family of k-planes in $\mathbb{P}^n$.

Warning 18.14. We are now shifting the index up by 1, so that we're looking at $\mathbb{G}(k, n) = G(k+!, n+1)$.

Definition 18.15 (The universal family of k-planes in $\mathbb{P}^n$)**.** We have an incidence correspondence

$$\Sigma := \{(\Lambda, p) : p \in \Lambda\} \subset \mathbb{G}(k, n) \times \mathbb{P}^n.$$

We have natural projections

$$\begin{array}{ccccc} & & \Sigma & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathbb{G}(k,n) & & & & \mathbb{P}^n \end{array} \tag{18.1}$$

The left map π_1 realizes $\mathbb{G}(k, n)$ as the family of k planes in $\mathbb{P}^n$, since the fiber of π_1 over a point $[\Lambda] \in \mathbb{G}(k, n)$ is the set of point $p \in \mathbb{P}^n$ with $p \in \Lambda$. That is, it is the subspace $\Lambda \subset \mathbb{P}^n$.

To show this is indeed a family, we need to check Σ is a variety.

Lemma 18.16. *We have that Σ is a variety in*

$$\mathbb{G}(k, n) \times \mathbb{P}^n \subset \mathbb{P} \bigwedge^{k+1} \Bbbk^{n+1}) \times \mathbb{P}^n.$$

Proof. We can realize Σ as the vanishing of a certain wedge product. Consider

$$\Phi := \{([\eta], [\nu]) : \eta \wedge \nu = 0\} \subset \mathbb{P} \bigwedge^{k+1} \mathbb{k}^{n+1}) \times \mathbb{P}^n.$$

Then,

$$\Sigma = \mathbb{G}(k, n) \times \mathbb{P}^n \cap \Phi,$$

where the intersection occurs in $\mathbb{P} \bigwedge^{k+1} \mathbb{k}^{n+1}) \times \mathbb{P}^n$. Since the intersection of two varieties is a variety, this universal family is a variety. □

We'll now use this universal family to construct other interesting varieties. To do this, we'll need two lemmas.

Lemma 18.17. *If* $\Phi \subset \mathbb{G}(k, n)$ *we have*

$$X_\Phi := \cup_{[\Lambda] \in \Phi} \Lambda \subset \mathbb{P}^n$$

is a projective variety.

Proof. Take

$$X_\Phi = \phi_1(\pi_2^{-1}\Phi).$$

□

Lemma 18.18. *If* $X \subset \mathbb{P}^n$ *is any projective variety, we have*

$$\Sigma := \{[\Lambda]\, \mathbb{G}(k, n) : \Lambda \cap X \neq \emptyset\} \subset \mathbb{G}(k, n).$$

Then, Σ *is a projective variety.*

Proof. Take $\Sigma = \pi_1(\pi_2^{-1}X)$. □

19. 3/4/16

Question 19.1. Given 4 general lines in 3-space how many lines meet all 4?

19.1. **Review of last time:** Recall that $\mathbb{G}(k, n)$ is a parameter space for the set of k-planes $\Lambda \subset \mathbb{P}^n$, i.e. there exists a projective variety

$$\Sigma = \{(\Lambda, p) : p \in \Lambda\} \subset \mathbb{G}(k, n) \times \mathbb{P}^n, \tag{19.1}$$

equipped with natural projections

$$\begin{array}{ccc} & \Sigma & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ \mathbb{G}(k, n) & & \mathbb{P}^n \end{array} \tag{19.2}$$

that forms a family via projection onto the first factor, in which the fiber over a point in $\mathbb{G}(k, n)$ is just the k-plane corresponding to that point.

Remark 19.2. Note that Λ means two different things in (19.1). The first time, it's a point in the Grassmannian, and the second time it's a k-plane. We will abuse notation and use Λ to mean both things.

To see that Σ is a variety, we observe that it is the intersection of $\mathbb{G}(k, n) \times \mathbb{P}^n$ with the set

$$\{([\omega], [v]) : w \wedge v = 0\} \subset \mathbb{P}^N \times \mathbb{P}^n,$$

where $\mathbb{P}^N = \mathbb{P}(\wedge^{k+1} K^{n+1})$.

As a consequence, have the following two lemmas.

Lemma 19.3. *Given any subvariety $\Phi \subset \mathbb{G}(k, n)$, the set*

$$\Psi = \bigcup_{[\Lambda] \in \Phi} \Lambda \subset \mathbb{P}^n$$

is a projective variety.

Proof. We have $\Psi = \beta(\alpha^{-1}(\Phi))$. □

We also have the opposite observation.

Lemma 19.4. *Given $X \subset \mathbb{P}^n$ a subvariety, let*

$$\mathcal{C}_X = \{[\Lambda] : \Lambda \cap X \neq \varnothing\} \subset \mathbb{G}(k, n).$$

Then $\mathcal{C}_X$ is a projective variety (called the variety of incident planes to X).

Proof. We have $\mathcal{C}_X = \alpha(\beta^{-1}(X))$. □

We'll use these ideas to answer the question from the beginning of class.

19.2. **Finding the equations of $\mathbb{G}(1, 3)$.** What's the first Grassmannian that's not a projective space? (This occurs when $k = 1$ or $n - 1$.) The first example is

$$\mathbb{G}(1, 3) = \{\text{lines in } \mathbb{P}^3\} \hookrightarrow \mathbb{P}(\wedge^2 K^4) = \mathbb{P}^5.$$

where $\wedge^2 K^4$ is a six dimensional vector space (if K^4 has basis $e_1, \ldots, e_4$, then $\{e_i \wedge e_j : i < j\}$ is a basis). It's not hard to write down the equations defining the Grassmannian. Think of $\wedge^2 K^4$ as skew-symmetric bilinear forms on K^4.

Lemma 19.5. *If* $q : V \times V \to K$ *skew-symmetric bilinear form (i.e. linear in each factor separately and picks up a minus sign under flipping the two entries) them there exists a basis for* V *(*$\cong K^n$*) such that*

$$q(v, w) = v^T A w$$

where A *is block diagonal with*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

along part of the diagonal and then zeros everywhere else, i.e.

$$q = e_1 \wedge e_2 + e_3 \wedge e_4 + \ldots + e_{2k-1} \wedge e_{2k}.$$

Remark 19.6. This is the analogue of the statement that symmetric bilinear forms can be diagonalized. As in that case, we will need the hypothesis that the characteristic of our field is not 2. The rank of a skew-symmetric bilinear form is defined to be the rank of the matrix A, so $2k$ above. Note that this shows the rank of a skew-symmetric bilinear form is always even.

Proof. If $q = 0$ we're done. If not, there exist $v, w \in V$ such that $q(v, w) = 1$. (There's some pair with non-zero inner product – scale so you get 1.) Take $e_1 = v$ and $e_2 = w$. Now restrict to the orthogonal complement,

$$\langle v, w \rangle^\perp = \{u \in V : q(v, u) = q(w, u) = 0\},$$

and repeat! If $q = 0$ on this subspace, you're done. Otherwise we can find vectors that pair to 1 and continue this process until $q = 0$ on what remains. ☐

Now back to our example, if V is 4-dimensional, for $\omega \in \wedge^2 V$ there exists a basis $e_1, \ldots, e_4$ for V such that either $\omega = e_1 \wedge e_2$ or $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$. In the first case, $[\omega] \in \mathbb{G}(1, 3) \subset \mathbb{P}(\wedge^2 V)$: it's the point corresponding to the plane spanned by e_1 and e_2. Note that in the case that $\omega = v \wedge w$ then $\omega \wedge \omega = 0$. However, when $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$, then $\omega \wedge \omega = 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0$. Thus, we can distinguish between the two cases by whether the self-wedge $\omega \wedge \omega$ is zero or not. Thus we have the following.

Lemma 19.7. *Given* $[\omega] \in \mathbb{P}(\wedge^2 V)$*,*

$$[\omega] \in \mathbb{G}(1, 3) \Leftrightarrow \omega \wedge \omega = 0.$$

In addition, $\mathbb{G}(1, 3) \subset \mathbb{P}(\wedge^2 V) = P^5$ *is a quadric hypersurface.*

Proof. We've got a bilinear form

$$\bigwedge^2 V \times \bigwedge^2 V \to \bigwedge^4 V = K \tag{19.3}$$

via "wedge together." Restricting to the diagonal we obtain a symmetric bilinear form, so $\omega \wedge \omega$ is a homogeneous quadratic polynomial on $\omega \in \bigwedge^2 V$. Hence, $\mathbb{G}(1,3) \subset \mathbb{P}(\bigwedge^2 V) = \mathbb{P}^5$ is a quadric hypersurface. □

19.3. **Subvarieties of $\mathbb{G}(1,3)$.** Fix a line L_0 in $\mathbb{P}^3$. The set of lines meeting L_0 as a subset of $\mathbb{G}(1,3)$, in fact a subvariety by the second lemma above. Suppose $L_0 = \langle \alpha, \beta \rangle$ (so L_0 corresponds to the point $\alpha \wedge \beta \in \bigwedge^2 V$) and say we have some other $L = v \wedge w$. When does $L = v \wedge w$ meet L_0? L meets L_0 if and only if $(v \wedge w) \wedge (\alpha \wedge \beta) = 0$. This is a linear equation on $v \wedge w$ because the map (19.3) is bilinear, so when you fix the second factor to $\alpha \wedge \beta$, you get a linear map. We conclude that $\mathcal{C}_{L_0}$ is a hyperplane section of $\mathbb{G}(1,3) \subset \mathbb{P}^5$.

Remark 19.8. In fact, $\mathcal{C}_{L_0}$ is the tangent hyperplane section at L_0. We haven't talked about this yet, but the tangent space is what you think intuitively. Since $\mathbb{G}(1,3)$ is a smooth quadric in $\mathbb{P}^5$, the tangent hyperplane section is a rank 4 quadric in $\mathbb{P}^4$, a cone over the smooth quadric in $\mathbb{P}^3$ with vertex $[L_0]$.

How about lines containing a point? Say $L = [\omega]$ and $p = [v]$. What does it mean to say $p \in L$? This is if and only if $v \wedge \omega = 0$. Now we're looking at a map

$$\bigwedge^2 V \times V \to \bigwedge^3 V$$

The conclusion is that $\Sigma_p = \{L : p \in L\}$ is a 2-plane inside $\mathbb{G}(1,3) \subset \mathbb{P}^5$.

Next: lines contained in a plane. (Note that lines meeting a plane is everything – any line and plane meet in $\mathbb{P}^3$). Abstractly, lines in a plane is the dual of $\mathbb{P}^2$. But how does it sit inside the Grassmannian in $\mathbb{P}^5$? A plane H in $\mathbb{P}^3$ corresponds to a 3-dimensional subspace W of my four-dimensional vector space V. The inclusion $W \hookrightarrow V$ gives rise to an inclusion $\bigwedge^2 W \hookrightarrow \bigwedge^2 V$, so

$$\Gamma_H = \{L : L \subset H\} = \mathbb{P}(\bigwedge^2 W) \hookrightarrow \mathbb{P}(\bigwedge^2 V) \simeq \mathbb{P}^5$$

is a linear subspace.

In conclusion

$$\Sigma_p = \{L : p \in L\}$$

and

$$\Gamma_H = \{L : L \subset H\}$$

are both two planes in $\mathbb{G}(1,3) \subset \mathbb{P}^5$. In fact, these are all of the 2-planes contained in the Grassmannian.

Lemma 19.9. *Every 2-plane on* $\mathbb{G}(1,3)$ *is one of these.*

Proof. On your homework :) □

A smooth quadric hypersurface in $\mathbb{P}^3$ is abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. It has two families of lines (called lines of the ruling), each parameterized by $\mathbb{P}^1$.

Our discussion about the Grassmannian tells us a that a quadric hypersurface in $\mathbb{P}^5$ has two families of 2-planes, each parameterized by $\mathbb{P}^3$.

Exercise 19.10. What about other quadrics? The answer requires some representation theory. There's something here if you feel like thinking about it.

19.4. **Answering our enumerative question.** Back to our question from the beginning of class:

Question 19.11. Given 4 general lines $L_1, \ldots, L_4 \subset \mathbb{P}^3$, how many lines meet all 4?

You might imagine you can visualize it ...but that's pretty hard. You could write out the equations. But you don't need to. Asking "how many" only requires that we know the type of polynomials. Recall that for each of these lines the locus

$$\mathcal{C}_{L_i} = \{\text{lines meeting } L_i\} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$$

is a hyperplane section $H_i \cap \mathbb{G}(1,3)$ of $\mathbb{G}(1,3)$. We can rephrase our question as what is the cardinality of

$$H_1 \cap H_2 \cap H_3 \cap H_4 \cap \mathbb{G}(1,3)?$$

Start by intersecting the hyperplanes: if they're linearly independent, we just get a line in $\mathbb{P}^5$. Now intersect that with $\mathbb{G}(1,3)$, which is the zero locus of a homogeneous quadratic polynomial. Restrict that equation to the line, so you're just asking for the zeros of a quadratic polynomial on $\mathbb{P}^1$, which we expect to be 2. Technically, you need to check that it has two distinct roots.

Remark 19.12. This is just the beginning of a notion in algebraic geometry called enumerative geometry, where we use parameter spaces to answer enumerative questions.

Remark 19.13. To use the phrase "general lines," we should know if the parameter space is irreducible. The answer is yes, although we didn't prove it. In fact, the Grassmannian $G(k, n)$ contains a dense open set isomorphic to affine space $\mathbb{A}^{k(n-k)}$. We'll talk about this more when we talk about dimension after break.

20. 3/7/16

20.1. **Plan.** This week is going to be about rational maps. It is something somewhat unique to algebraic geometry. This is Chapter 7 in the text. If we have time, we'll look at some examples from Chapter 8. After break, we'll start right in with Chapter 11, which includes the notion of the dimension of a variety. (The dimension of a variety is what you think it is, but unfortunately the actual definition is not. In the 19th century, people just looked at varieties and said the dimension was obvious, but in the 20th century, we realized this definition was more tricky.)

20.2. **Rational functions.** This is going to be an awkward talk because the language we use is inexact and not very uniform. Let $X \subset \mathbb{A}^n$ be an irreducible affine variety. The basic object we want to look at is quotients of regular functions on X. We're going to refine the following definition throughout the lecture.

Definition 20.1 ("Definition")**.** A rational function on X is a function locally expressible as a ratio f/g where f and g are regular functions.

There's a problem with this definition: the objects being described are not functions! Say $X = \mathbb{A}^2$ and $h(x, y) = y/x : \mathbb{A}^2 \to \mathbb{A}^1$. As a map to $\mathbb{A}^1$ it's not defined where $x = 0$. We can enlarge the target from $\mathbb{A}^1$ to $\mathbb{P}^1$, but it's still not defined when $x = y = 0$. At this point, there is no way to extend the map. You can approach the origin from any line through the origin and the function is constant along these lines, but the limit is different depending on which direction you come from.

It's not completely unreasonable to think of these as functions though, because they're defined on dense open sets. How do we tell if two of these not actually functions are equal? We could say $f/g = f'/g'$ if $fg' = f'g$. This is what the fraction field will do for us. Let $I(X) \subset K[x_1, \ldots, x_n]$ be the ideal of functions vanishing on X, and let $A(X) = K[x_1, \ldots, x_n]/I(X)$ be the coordinate ring of X. If X is irreducible then $A(X)$ is an integral domain, so we can form its fraction field.

Definition 20.2. Let $K(X)$ be the field of fractions of $A(X)$. This is called the function field of X. A rational function on X is an element of $K(X)$.

These are not quite functions, but they are on nonempty open subsets of X, which is dense since X is irreducible.

Remark 20.3. To get between these two definitions, we have to remove the word "locally." This follows from the Nullstellensatz. We put it in there though, so we could extend it directly to projective varieties.

Now suppose $X \subset \mathbb{P}^m$ is a projective variety. We can also realize $K(X)$ as the function field of any open affine subset.

Remark 20.4. This requires checking that this doesn't depend on the choice of open affine subset.

Alternatively, let $I(X) \subset K[x_0, \ldots, x_m]$ be the homogeneous ideal and $S(X) = K[x_0, \ldots, x_m]/I(X)$ the homogeneous coordinate ring. $S(X)$ is a graded ring. When we take its field of fractions, we still have a notion of degree: degree of numerator minus degree of denominator, so it is again a graded ring (but also with negative degrees, so graded by $\mathbb{Z}$). If we want to get functions, we need quotients of homogeneous polynomials of the same degree (because the result must be well defined when we rescale).

Definition 20.5. For a projective variety X, the function field $K(X)$ is the zeroth graded piece of the field of fractions of $S(X)$.

This all motivates our real definition of rational function.

Definition 20.6. A rational function on a variety X is an equivalence class of pairs (U, f) where $U \subset X$ is a nonempty (Zariski) open set and f is a regular function on U, with the equivalence relation

$$(U, f) \sim (V, g) \Leftrightarrow f = g \text{ on } U \cap V.$$

You can see why we didn't start class with this. This subsumes all the definitions. Put simply, a rational function is a regular function on a dense open subset, but to deal with the problem of when two rational functions are equal, we need this fancy equivalence relation.

Now that we've got rational functions, we can define rational maps.

Definition 20.7. A rational map $f : X \dashrightarrow Y$ is an equivalence class of pairs (U, f) with $f : U \to Y$ regular, where two are equivalent if they agree on the intersection of their domains.

Remark 20.8. We used a dashed arrow for rational functions to distinguish them from regular functions.

Remark 20.9. We're still assuming X is irreducible. If we want to extend this to reducible varieties, we need to add the condition that open sets U are dense. But then you don't get a field: you get a direct sum of function fields on the irreducible components. Basically, its the data of a rational function on each irreducible component.

This is really special to algebraic geometry: in most geometric categories you can't just invert any function. Here, we basically can because the locus where a function vanishes is so small.

Definition 20.10. We say X and Y are birational (birationally isomorphic) if there exist rational maps $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow X$ which are inverse to one another.

Since rational maps are not really maps, you can't just compose them, but it'll be defined if you restrict to an appropriate open subset. We're asking that the composition is the identity there.

Prof. Harris: What's the simplest variety you know?
Aaron Slipper: The empty set.
Prof. Harris: I was looking for affine space.

Definition 20.11. We say X is rational if X is birationally isomorphic to affine space $\mathbb{A}^n$, equivalently if $K(X) = K(x_1, \dots, x_n)$.

Example 20.12. Say we have $Q = V(XY - ZW) \subset \mathbb{P}^3$ a smooth quadric surface. Let $p = [0, 0, 0, 1] \in Q$ and consider the projection map $\pi_p : Q \dashrightarrow \mathbb{P}^2 = V(W)$. This sends $q \mapsto \overline{p, q} \cap \mathbb{P}^2$ or in coordinates, $[X, Y, Z, W] \mapsto [X, Y, Z]$. This is a rational map because it's not defined at the point p, where all three of X, Y, Z vanish. (And it cannot be extended to a regular map defined at p.) Let $L = V(X, Z)$ and $M = V(Y, Z)$ be the pair of lines on Q that meet p. The map is generically one-to-one, but it collapses L and M. It sends $L \to [0, 1, 0]$ and $M \to [1, 0, 0]$. It's not an isomorphism (very far from it – not defined everywhere, not one-to-one), but it is a birational isomorphism. Here's its rational inverse, $\phi : \mathbb{P}^2 \dashrightarrow Q$ is defined by $[X, Y, Z] \mapsto [X, Y, Z, \frac{XY}{Z}] = [XZ, YZ, Z^2, XY]$.

Contrast this example with the projection of a conic onto a line in $\mathbb{P}^2$. In this case, we actually could extend it to a regular map, and a plane conic is isomorphic to $\mathbb{P}^1$. Example 20.12 is the one-dimension up version, and now we can't extend the projection to a regular map.

In this case, $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$ (Proof: any two curves in $\mathbb{P}^2$ intersect, but there are lines on $\mathbb{P}^1 \times \mathbb{P}^1$ that do not meet, e.g. $[0,1] \times \mathbb{P}^1$ and $[1,0] \times \mathbb{P}^1$.)

21. 3/9/16

21.1. **Rational Maps.**

Definition 21.1. A rational map $X \dashrightarrow Y$ is an equivalence class of pairs (U, f_0) with $U \subset X$ open dense and $f_0 : U \to Y$ a regular map.

The equivalence relation is given by $(U, f_0) \sim (V, g_0)$ if $f_0|_{U \cap V} = g_0|_{U \cap V}$.

Remark 21.2. There is very often a disconnect between the way people work in the field with a given object and the definition of that object. As a key example, consider the notion of a rational map $X \dashrightarrow Y$. They thought of this as "a map given by $[f_0, \ldots, f_n]$ with $f_i \in K(X)$." The more standard definition is the above definition.

Lemma 21.3. *For any rational map $f : X \dashrightarrow Y$ there is a maximal representative (U, f_0), meaning for every other pair in this equivalence class (V, g_0) we have $V \subset U$.*

Proof. If we have maps defined on U and V then we can define a map on $U \cup V$ by defining it to agree with f_0 for any point in U and g_0 for any point in V. $\square$

Definition 21.4. Retaining the terminology of Lemma 21.3, we call U the **domain of definition** of f and $X \setminus U$ the **indeterminacy locus**.

Example 21.5. Let $L \subset \mathbb{P}^2$ and $p \in \mathbb{P}^2$ be a point not on L. If we have a projection

$$\begin{aligned} \pi_p : \mathbb{P}^2 &\to L \cong \mathbb{P}^1 \\ q &\mapsto \overline{pq} \cap L \\ [X, Y, Z] &\mapsto [X, Y] \end{aligned}$$

Here, the domain of definition is $\mathbb{P}^2 \setminus \{p\}$.

21.2. **Operations with rational maps.**

Given a rational maps $f : X \dashrightarrow Y$, we will define

(1) composition of rational maps
(2) the image of f
(3) the preimage $f^{-1}(Z)$ of $Z \subset Y$
(4) the graph of f.

Definition 21.6. Let $f : X \dashrightarrow Y$ be a birational morphism of varieties. Take (U, f_0) a representative of an equivalence class of f, with $f_0 : U \to Y$, $U \subset X$. Then, we define

$$\Gamma_{f_0} \subset U \times Y \subset X \times Y.$$

Then, define Γ_f to be the closure of Γ_{f_0} in the Zariski topology on $X \times Y$.

Exercise 21.7. Show that the graph is independent of the choice of representative of an equivalence class of a birational map. *Hint:* First do this for irreducible varieties. For this irreducible case, note that the closure of two dense sets in the product will agree.

Definition 21.8. Let $f : X \dashrightarrow Y$ and let Γ_f be its graph with

(21.1)
$$\begin{array}{ccc} & \Gamma_f & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ X & & Y. \end{array}$$

Then, the **image** of f is by definition $\beta(\Gamma_f)$. For $Z \subset X$, the **image** of Z is

$$f(Z) = \beta(\alpha^{-1}(Z))$$

For $W \subset \mathbb{P}^n$, the **preimage** of W is

$$f^{-1}(W) = \alpha(\beta^{-1}(W)).$$

Example 21.9. Let $\pi_p : \mathbb{P}^2 \dashrightarrow L$ be the projection away from p. Then, the image of p is all of L. Be warned, the image of a point in this case is a line!

Definition 21.10. Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ and suppose there exists pairs $(U, f_0), (V, g_0)$ with $U \subset X, V \subset Y$. Suppose further that $f_0(U) \not\subset Y \setminus V$. Then, we define the composition $g \circ f$ to be the equivalence class of the pair

$$\left(f_0^{-1}(V), g_0 \circ f_0\right).$$

Definition 21.11. We say two varieties X and Y are **birational** or **birationally isomorphic** if there exists rational maps $f : X \dashrightarrow Y, g : Y \dashrightarrow X$ if $f \circ g, g \circ f$ are both defined and are the identity on their domain of definition.

That is, there exist dense open subsets $U \subset X, V \subset Y$ and equivalence classes $(U, f_0), (V, g_0)$. with an isomorphism $U \cong V$ defined by f_0 and g_0.

We say X is **rational** if X is birational to $\mathbb{P}^n$.

Fact 21.12. Suppose X and Y are both irreducible. In fact, X is birational to Y if and only if $K(X) \cong K(Y)$, where $K(Z)$ denotes the field of rational functions on Z.

Example 21.13 (Quadric hypersurfaces are rational)**.** Suppose $Q \subset \mathbb{P}^{n+1}$ is a quadric hypersurface of maximal rank. That is, $\text{rk}(Q) = n+2$. For $p \in Q$, define the map

$$\pi_p : Q \dashrightarrow \mathbb{P}^n$$

to be the projection away from p.

This projection is a birational map.

Example 21.14. Show this is birational. *Hint:* Choose a convenient quadric and then writing out the map in coordinates.

The map is not defined at p. On the quadric surface in $\mathbb{P}^3$, it is also not injective on the two lines passing through p. But, for L, M the two lines through p, and $q = \pi_p(L), r = \pi_p(M)$, we obtain an isomorphism $Q \setminus (L \cup M) \cong \mathbb{P}^2 \setminus \overline{qr}$. In general, we obtain an isomorphism between the quadric minus the lines through the point we are projecting from, and the image of that open set in $\mathbb{P}^n$.

Question 21.15. We just saw quadric hypersurface are rational. Are cubic hypersurfaces rational?

In fact, a cubic curve in $\mathbb{P}^2$ is not rational if and only if it is smooth.

Exercise 21.16 (Tricky Exercise)**.** Show that $x^3 + y^3 + z^3 = 0$.

Here is a summary of which cubic surfaces are rational. We assume all varieties are smooth (even though we haven't defined smoothness yet).

(1) Cubic curves in $\mathbb{P}^2$ are not rational. This was probably shown in the early 19th century.
(2) Cubic surfaces in $\mathbb{P}^3$ are rational. (In fact, Miles Reid's introductory algebraic geometry textbook treats this as the lynchpin of the book.)
(3) The cubic threefold in $\mathbb{P}^4$ was not rational as was shown in 1972 by Clemens and Griffiths. This proof uses hodge theory.
(4) The cubic fourfold in $\mathbb{P}^5$ is currently unknown. The current belief is that some are rational and some are not. We know that some cubic fourfolds are rational, but it is not known

whether there are any ones which are not rational, or what the locus of smooth ones are.

(5) When n is even, there are examples of rational cubic n-folds.
(6) When n is odd, it is not known whether there exist rational cubic n-folds or whether there are irrational such n-folds.

Remark 21.17. In the first two cases, if one has a family of curves or surfaces, then either all are rational or all or all are not rational. In higher dimensions, it is not known whether the rationality condition is open, closed, both, or neither.

22. 3/11/16

Today we'll talk about

- Calculus
- Blowing up
- Examples

22.1. **Calculus.** We talked last time about the role of rationality in the development of the subject. The whole subject got a tremendous boost from analysis in the beginning of the 19th century.

Back to the 18th century, people were just learning how to use calculus and determining which functions you could integrate. For example,

$$\int \frac{dx}{\sqrt{x^2-1}} \tag{22.1}$$

which they figured out how to do by trigonometric substitution. The next thing to try is

$$\int \frac{dx}{\sqrt{x^3-1}}. \tag{22.2}$$

But the whole program hit the wall. They couldn't find a function whose derivative was the integrand and worse, they couldn't understand why they couldn't integrate it.

We can think of the integral in (22.1) as $\int \frac{dx}{y}$ on the curve $y^2 = x^2 - 1$. The crucial observation is that this is a rational curve. It's birationally isomorphic to $\mathbb{P}^1$ via projection away from any point on the curve. It has a rational parameterization

$$x = \frac{1+t^1}{1-t^2} \qquad \text{and} \qquad y = \frac{2t}{1-t^2}.$$

In the reverse direction, $t = \frac{y}{x+1}$. Thus, our integral becomes

$$\int \frac{dx}{y} = \int R(t)dt \qquad R \in \mathbb{C}(t),$$

and we know how to evaluate integrals of rational functions.

The problem with $\int \frac{dx}{\sqrt{x^3-1}}$ is that the curve $y^2 = x^3 - 1$ is not rational. The fact that this curve cannot be parameterized by rational functions in one variable is key to understanding why our techniques fail.

In fact, going a little further, we'll see that we shouldn't be able to integrate (22.2). Taking the closure in projective space, $V(Y^2 - X^2 + Z^2) \subset \mathbb{P}^2_{\mathbb{C}}$ is a sphere. When we picture it as a hyperbola, we're looking at the complement of two points of the sphere. If we homogenize the cubic, $V(Y^2Z - X^3 + Z^3)$ we get a torus! On a sphere, integrals are path independent. On a torus, however, the path between the two points matters. In fact there's a $\mathbb{Z}^2$ of possible values depending how many times you go around the generators of homology of the torus. If we look at $\int_p^q \omega$ as a function of q, it's a doubly periodic function in q. In the 18th century, they didn't know any doubly periodic functions in the complex plane (none of the elementary functions are). Once they had this picture, they could see that the integrand could not be expressed as the derivative of an elementary function. This led to the introduction of the Weierstrass $\wp$ function. This was also the first appearance of topology in the subject, and it sparked a great deal of mathematical development throughout the 19th century. In the context of this course, it illustrates how the rationality of a curve can be crucial.

Exercise 22.1. Prove that $y^2 = x^3 - 1$ is not rational. Hint: show there do not exist non-constant rational functions $a(t), b(t) \in K(t)$ such that $a(t)^2 = b(t)^3 - 1$.

Exercise 22.2. Prove that topologically, the conic is a sphere and the smooth cubic is a torus.

Remark 22.3. One way to prove these is to develop the notion of the genus of a curve. This requires more focus on algebraic curves than we'll have in this course.

22.2. **Blow up of $\mathbb{P}^2$ at a point.** Back to our example of projection from a point $p \in \mathbb{P}^2$. Pick some line L not containing p. We can choose coordinates so that $p = [0, 0, 1]$ and $L = V(Z)$. Consider $\pi_p : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by $p \mapsto \overline{p, q} \cap L$. In coordinates, it sends $[X, Y, Z] \mapsto [X, Y]$.

The key object to associate to a rational map is its graph. The graph Γ of π_p is the closure of the graph on an open subset where the map is defined. We'll describe it as

$$\begin{aligned}\Gamma &= \{(q,r) \in \mathbb{P}^2 \times L : p, q, r \text{ collinear}\} \\ &= \{([X,Y,Z],[A,B]) : BX = AY.\}\end{aligned}$$

Notice that this is a closed subvariety of $\mathbb{P}^2 \times L$. In the second way its the vanishing of a bihomogeneous polynomial of bidegree $(1,1)$. What does this look like? Look at the projection onto the first factor $\alpha : \Gamma \to \mathbb{P}^2$. The fiber of α over a point $q \in \mathbb{P}^2$ is a single point if $q \neq p$ but if X, Y are both zero, the equation doesn't impose any condition, so the fiber is $\mathbb{P}^1$ over $q = p$.

Definition 22.4. We call $\Gamma \to \mathbb{P}^2$ the blow up of $\mathbb{P}^2$ at the point p.

Remark 22.5. This is the subject of a lot of stories. Apparently, some students made t-shirts which said "We blew up the plane" to commemorate a summer conference once. There's a more-detailed correct version of the story that Professor Harris might tell at some point.

The map π_p is constant along every line through p, but assumes different values on different lines. What this means is that the graph is one-to-one over $\mathbb{P}^2$ away from p and over p assumes all values of slopes of lines though p. It's as if we took all the lines in the plane through p and made them disjoint. It's just like a spiral staircase where the stairs come out from a central axis and the top of the stairs it's the same as the bottom. Whenever you hear the word blow up, you should think of this picture.

22.3. **Blow ups in general.** More generally, suppose we start with any variety $X \subset \mathbb{P}^m$ and $Z \subset X$ a closed subvariety. Find homogeneous polynomials $F_0, \dots, F_n$ of the same degree such that the saturation of the ideal they generate is $I(Z)$. In other words, $F_0, \dots, F_n$ cut out Z scheme theoretically. Construct the rational map $X \dashrightarrow \mathbb{P}^n$ by $[F_0, \dots, F_n]$. If we want a regular map, we have to insist that these polynomials don't all simultaneously vanish, or extend the map over the locus where they do. For a rational map, we don't have to do this. Let Γ be the graph of ϕ sitting inside $X \times \mathbb{P}^n$. Note that the map $\Gamma \to X$ is an isomorphism over $X \setminus Z$: if the F_i's don't all vanish there's a unique point above that point on the graph, namely the image of that point.

Definition 22.6. $\Gamma \to X$ is called the blow up of X along Z.

Remark 22.7. The blow up is not just the variety Γ, but Γ together with a map to X.

Remark 22.8. If you choose a different set of generators F_i than you'll get something isomorphic over X, meaning there exists an isomorphism $\Gamma \cong \Gamma'$ that commutes with the maps to X.

Exercise 22.9. Describe the blow up of $\mathbb{P}^3$ along a line. Hint: project away from this line in $\mathbb{P}^3$ to a complementary line and think about the graph.

Here's a result that illustrates why blow ups are so fundamental.

Theorem 22.10. *If* $S \dashrightarrow T$ *is any birational isomorphism between smooth surfaces* S *and* T, *then* ϕ *factors into a sequence of blowups at points.*

What this says is blow ups at points generate all birational isomorphisms of surfaces. So if you understand the geometry of this simple example of a blow up at a point, then you understand all birational maps between surfaces.

22.4. **Example: join of a variety.** Suppose we have varieties $X, Y \subset \mathbb{P}^n$ disjoint. Then

$$J = \bigcup_{p \in X, q \in Y} \overline{p, q}$$

is a projective variety. To see this, look at the map $X \times Y \to \mathbb{G}(1, n)$ given by $(p, q) \mapsto \overline{p, q}$. Then J is the union of all lines parameterized by some subvariety of the Grassmannian, so it's a variety.

What if X and Y meet? Then we get a rational map $s : X \times Y \dashrightarrow \mathbb{G}(1, n)$. This rational map has a well-defined image and we define the join of X and Y to be

$$J(X, Y) = \bigcup_{[L] \in \mathrm{Im}(s)} L.$$

Say $X \subset \mathbb{P}^n$ is any subvariety. Take $X = Y$. We still get a rational map to the Grassmannian, so we can form $J(X, X)$. This is called the secant variety. It's the closure of all secant lines to X.

23. 3/21/16

23.1. **Second half of the course.** We're starting the second part of the book today. Now that we have a basic vocabulary, we'll be able to prove some theorems about varieties. For that, we'll need to describe certain attributes of varieties. Today, we'll discuss dimension. (Now words like "curve" and "surface" will have content, meaning one-and two-dimensional varieties.) Coming up, we'll talk about degree, the Hilbert polynomial, and the Hilbert function. These are all global attributes of $X \subset \mathbb{P}^n$. We'll also look at local properties, such as smoothness, tangent spaces, and tangent cones. A large part of what we'll do is look for appropriate algebraic conditions that allow us to extend these notions to algebraic varieties over arbitrary fields.

The textbook is rushed – it doesn't build up a complete, precise, logical framework. This reflects historical approaches to algebraic geometry, but it can be frustrating in the modern day, especially if you're supposed to be proving things on problem sets. But after a bumpy beginning, we'll develop a nice roster of theorems that allow us to further describe varieties arising from examples we saw at the beginning of the course.

If you find yourself looking at a problem set and you're not exactly sure what we're asking for in the question, what you need to calculate, or how to get started, please ask! Ask Aaron, ask Hannah, or ask Professor Harris. We're all happy to talk about this stuff. It could be really helpful to get a sense of what the question is asking for before you start banging your head over it.

23.2. **Dimension historically.** We'll start off assuming that X is an irreducible projective variety in $\mathbb{P}^n$. (Although n shouldn't matter – a variety should have a dimension which is the same regardless of how its embedded in projective space.) In the 19th century, the problem of the dimension of a variety didn't even occur to people! It seemed "obvious" what dimension should be. Back then their definition was: "A variety is said to have dimension k if it contains an infinity to the k of points." We can interpret this as saying that locally a point on the variety is specified by k parameters varying freely. This corresponds to our modern notion of a manifold: we're saying that, at least locally, we can describe a point by specifying k variables, like in a coordinate chart. In fact, you can make this into a definition if you're working over the complex numbers.

Consider $X/\mathbb{C}$ with the classical topology.

Proposition 23.1. *There exists $U \subset X$ open, dense such that U is a complex manifold.*

We won't prove this now. We can use it to make the following definition though.

Definition 23.2. The dimension of X to be the dimension of U as a complex manifold.

Note that this doesn't work over the reals. Over the reals, $V(x^2 + y^2)$ is a single point, so this would say the dimension is zero, but we want $V(x^2+y^2)$ to be a plane curve, something of dimension 1. This is probably the closest definition to the 19th century conception of dimension. But it's no good: it involves this proposition we haven't proved, and it just shifts the problem off to complex geometry. We want something we can work with algebraically.

23.3. **Useful characterization of dimension.** Let's start with something we all agree on: the dimension of $\mathbb{P}^k$ should be k. Now I'd like to say if I have a finite to one map onto $\mathbb{P}^k$, the starting variety should also have dimension k:

Definition 23.3 ("Definition"). We say X has dimension k if there exists a finite surjective map $f : X \to \mathbb{P}^k$.

Remark 23.4. Maps are always regular, unless otherwise specified.

You might think this is bad for a couple reasons:

(1) If I have a variety X, how do I know such a map exists?
Solution: We'll show how to construct one now. Choose any $p \notin X$, $H \simeq \mathbb{P}^{n-1} \subset \mathbb{P}^n$ such that $H \not\ni p$, and project from p. The map $\pi_p : X \to \mathbb{P}^{n-1}$ is finite. (If we have some line joining p to a point on X, since $p \notin X$ there's a polynomial vanishing on X that doesn't vanish on the entire line and it can only have finitely many roots on that line.) Now just repeat the process until the map is surjective.

(2) Why couldn't there be more than one map: one to $\mathbb{P}^k$ and one to $\mathbb{P}^\ell$?
Solution: We'll address this later in the lecture to show that it is well defined.

We said we could arrive at a finite surjective map $f : X \to \mathbb{P}^k$ via a sequence of projections from points. We can also think of this as follows. Given $X \subset \mathbb{P}^n$, we can find (for some k) an $(n-k-1)$-plane $\Lambda \subset \mathbb{P}^n$ with $\Lambda \cap X = \varnothing$, i.e. $\pi_\Lambda : X \to \mathbb{P}^k$ is surjective. To say this map is surjective is to say that every $(n-k)$-plane containing Λ must meet X. This leads us to another characterization of dimension.

Definition 23.5. The dimension of X is k if there exists an $(n-k-1)$-plane $\Lambda \subset \mathbb{P}^n$ such that $\Lambda \cap X = \varnothing$ but every $(n-k)$-plane $\Gamma \supset \Lambda$ does meet X, i.e.

$$\dim X = \max\{k : \exists \Lambda \cong \mathbb{P}^{n-k-1} \subset \mathbb{P}^n \text{ with } \Lambda \cap X = \varnothing\}$$

Remember that the locus of planes meeting X is a closed subvariety of the Grassmannian. If there exists an $(n-k-1)$ plane disjoint from X, then the locus of $(n-k-1)$-planes meeting X is a proper subvariety, so a general $(n-k-1)$-plane is disjoint from X. Thus, we can say X has dimension k in $\mathbb{P}^n$ if a *general* $(n-k-1)$-plane is disjoint from X but *every* $(n-k)$-plane meets X.

This is a useful characterization, but still not a great definition because its not clear that the dimension is well-defined.

23.4. **The official definition.** To take care of this, we'll come up with an equivalent, but more opaque definition, which more clearly well-defined.

Observe that if $f : X \to \mathbb{P}^k$ is finite and surjective, the pull back map

$$f^* : K(\mathbb{P}^k) \to K(X)$$

is a finite algebraic extension. What this says is that $K(X)$ has transcendence degree k over K (the field of scalars).

Definition 23.6. The dimension of X is the transcendence degree of $K(X)$ over K.

Remark 23.7. Transcendence degree is supposed to be covered in 123, but sometimes it's not. Furthermore, some people in the class haven't taken 123. The point is that transcendence degree of a field extension is well-defined, so this reassures us that dimension is well-defined.

Remark 23.8. There are also definitions of dimension of a ring, in terms of the maximal length of chains of prime ideals or transcendence degree of field of fractions. If you're interested take a look at Atiyah-MacDonald.

Here's the geometric version of this: observe that if $Y \subset X$ is a proper closed subvariety, then $\dim Y < \dim X$. If $Y = H \cap X$ where $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is a hyperplane not containing X, then $\dim Y = \dim X - 1$. So yet another way to characterize dimension is in terms of a maximal increasing chain of proper subvarieties.

23.5. **Non-irreducible, non-projective case.** Now suppose X is an irreducible, quasi-projective variety. This means X is an open subset of its closure $\overline{X}$. We just define $\dim X = \dim \overline{X}$.

If $X = X_1 \cup \cdots \cup X_m$ is reducible with irreducible components X_i, we define

$$\dim X = \max(\dim X_i)$$

We can also give a definition of local dimension: for $p \in X$, define
$$\dim_p X = \max\{\dim X_i : X_i \ni p\}.$$
This is the same as the dimension of the local ring at p.

The transcendence degree of the function field is the standard definition. However, the characterizations in the first part of lecture are most useful in practice. No one really calculates the transcendence degree of the function field. It's necessary to make sure it's well defined, but we'll use the other characterizations in practice.

Example 23.9. Suppose $X = V(F) \subset \mathbb{P}^n$ is a hypersurface, where F is a polynomial of degree $d \geq 1$. Then $\dim X = n - 1$. You can see it from the characterization in terms of k-planes: if F is non-zero, then it's non-zero somewhere, so that's a point not in X. Therefore, a general point is disjoint from X. On the other hand, there cannot be a line disjoint from X. If we restrict F to that line, fundamental theorem of algebra tells us the polynomial vanishes somewhere on that line. Hence, every line meets X.

24. 3/23/16

Question 24.1. Let $F(X, Y, Z, W)$ be a general homogeneous polynomial of degree d and $S = V(F) \subset \mathbb{P}^3$ the corresponding surface. Does S contain a line?

24.1. Homework.

The next homework assignment will due next Friday, April 1. We'll have one assignment each week thereafter, for a total of 5 more assignments, with the last due April 29 (two days after the last class). The course will be graded entirely on homework.

Please talk to Aaron or Hannah if you have any questions about what is involved on homework!

24.2. Calculating dimension.

Today, we'll see how to actually calculate the dimension of some varieties. We'll start off with a basic theorem (which we won't prove yet) and then go on to see how it's used.

Theorem 24.2. *Let X be a projective irreducible variety and $f : X \to Y = f(X) \subset \mathbb{P}^n$. (The image Y must also be an irreducible projective variety.) For $q \in Y$, let $\lambda(q) = \dim f^{-1}(q) \subset X$. Then λ is upper-semicontinuous in the Zariski topology (meaning the locus where $\lambda \geq m$ is a closed set for all m). Furthermore, if $\lambda_0 = \min \lambda(q)$, then $\dim X = \dim Y + \lambda_0$.*

Remark 24.3. You should think of upper-semicontinuous as saying the fiber dimension can jump up on a closed set but can't jump down. Note that this implies the minimum fiber dimension is achieved on an open set (since the locus where fibers have higher dimension is a proper closed subset).

Our first example where λ is non-constant is a blow up. For example, suppose

$$X = \mathrm{Bl}_p\mathbb{P}^n \to \mathbb{P}^n$$

Recall $\mathrm{Bl}_p\mathbb{P}^n$ is the graph of the projection map $\pi_p\mathbb{P}^n \to \mathbb{P}^{n-1}$. We can write

$$X = \{(q, r) \in \mathbb{P}^n \times \mathbb{P}^{n-1} : p, q, r \text{ colinear}\}.$$

If q is not p, then the fiber over q is a single point (so dimension 0) while over p the fiber is a copy of $\mathbb{P}^{n-1}$ (so has dimension $n-1$).

Here's an example where this fails for manifolds. Take the ordinary sphere

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and project down to $[-1, 1]$ by taking the x coordinate. Over $(-1, 1)$ the fiber is a circle, but over ± 1 you only get a single point.

This isn't a counter example to our theorem though, because that was over the reals: it doesn't reflect what happens over the complex numbers. If we think of this equation over $\mathbb{C}^3$ then the fiber over ± 1 is given by $y^2 = z^2$, i.e. $z = \pm iy$, which is the union of two lines. You just don't see them in the real picture.

Remark 24.4. This illustrates why working over $\mathbb{C}$ is nicer, but it's also harder to visualize.

Corollary 24.5. *Say X is projective and $f : X \to Y \subset \mathbb{P}^n$ is a surjection onto some irreducible variety Y. If every fiber of f is irreducible of dimension k, the X is irreducible of dimension $\dim X = \dim Y + k$.*

Proof. Suppose we could write $X = X_1 \cup \cdots \cup X_m$ with X_i irreducible. Let $f_i = f|_{X_i}$ and let $\lambda_i(q)$ be the dimension of the fiber $\dim f_i^{-1}(q) \subset X_i$. We know that

$$k = \lambda(q) = \max\{\lambda_i(q)\}$$

so there exists some i such that $\lambda_i(q) = k$ on an open subset of Y. We said all the fibers have dimension k, so in fact $\lambda_i(q) = k$ for all q. Since $f^{-1}(q)$ is irreducible of dimension k, this implies $f_i^{-1}(q) = f^{-1}(q)$ for all q. □

24.3. **Dimension of the Grassmannian.** Let's determine the dimension of the Grassmannian

$$G(k,n) = \{k-\text{dimensional v. subspaces } \Lambda \subset V \simeq K^n\}.$$

Choose some $\Gamma \simeq K^{n-k} \subset V$ and look at the open subset

$$U_\Gamma = \{\Lambda \in G(k,n) : \Lambda \cap \Gamma = 0\}$$

We can choose a basis $e_1, \ldots, e_n$ for V so that $\Gamma = \langle e_{k+1}, \ldots, e_n \rangle$. Any $\Lambda \in G$ can be represented as the row space of a $k \times n$ matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

With this description,

$$U_\Gamma = \{\Lambda : \text{ first } k \times k \text{ submatrix is nonsingular}\}$$

For any $\Lambda \in U_\Gamma$ there is a unique matrix representative of the form

$$\begin{pmatrix} I_k & A \end{pmatrix}$$

where I_k is $k \times k$ identity matrix and A is $k \times (n-k)$ matrix. Hence,

$$U_\Gamma \cong \mathbb{A}^{k(n-k)}$$

so the Grassmannian is irreducible of dimension $k(n-k)$. In fact this is the complement of a hyperplane section.

24.4. **Dimension of the universal k-plane.** Now I want to think of the Grassmannian as linear spaces in projective spaces. This means we shift indices by one. Recall the universal k-plane

$$\Sigma = \{(\Lambda, p) : p \in \Lambda\} \subset \mathbb{G}(k,n) \times \mathbb{P}^n,$$

with its two projections

$$\begin{array}{ccc} & \Sigma & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{G}(k,n) & & \mathbb{P}^n \end{array} \tag{24.1}$$

First let's think of Σ as a family over $\mathbb{G}(k,n)$ via π_1. The fibers of π_1 are all dimension k (the fiber over a point is just the k-plane specified by that point). Thus our theorem says

$$\dim \Sigma = \dim \mathbb{G}(k,n) + k = (k+1)(n-k) + k$$

(because $\mathbb{G}(k,n) = G(k+1, n+1)$ – sorry for switching back and forth).

What are the fibers of the second projection? For $p \in \mathbb{P}^n$,

$$\pi_2^{-1}(p) = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \in p\} \simeq G(k,n) = \mathbb{G}(k-1,n-1)$$

On the vector space side, we're looking at $k+1$ dimensional vector spaces that contain a vector v corresponding to p. That's the same as a k dimensional vector spaces in the quotient K^{n+1}/v.

Remark 24.6. Note this is a sub-Grassmannian – not just a plane. For the first non-trivial example, look at 2-planes in $\mathbb{P}^4$ that contain a point. This corresponds to lines in a $\mathbb{P}^3$, so we get a copy of $\mathbb{G}(1,3)$.

Using this we can compute the dimension of Σ again:

$$\dim \Sigma = \dim \mathbb{P}^n + k(n-k) = n + k(n-k).$$

Aaron Slipper: Ah, so

$$(k+1)(n-k) + k = n + k(n-k).$$

Prof. Harris: Yes, but I think there's a more direct argument for that.

24.5. **Dimension of the variety of incident planes.** Suppose we have $X \subset \mathbb{P}^n$ of dimension l. Recall the variety of incident k-planes

$$\mathcal{C}_X = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \cap X \neq \varnothing\}.$$

What is the dimension of $\mathcal{C}_X$?

We can realize $\mathcal{C}_X$ as $\pi_1(\pi_2^{-1}(X))$. (This is how we saw it was a closed subvariety of the Grassmannian.) The same set-up allows us to say what its dimension is. We have $\pi_2^{-1}(X) \to X$ with each fiber isomorphic to $\mathbb{G}(k-1,n-1)$, so $\pi_2^{-1}(X)$ irreducible of dimension $l + k(n-k)$. Now we want the image under π_1. We need a little lemma for this.

Lemma 24.7. *Assume* $k + l < n$. *Then a general fiber of* $\pi_1 : \pi_2^{-1}(X) \to \mathbb{G}(k,n)$ *is finite (in fact a single point), i.e. for a general* k*-plane* Λ *meeting* X, *the intersection* $\Lambda \cap X$ *is finite.*

Since π_1 is finite, we can conclude that $\dim \mathcal{C}_X = l + k(n-k)$. This has codimension $n-k-l$ in $\mathbb{G}(k,n)$. In the case $k+l \geq n$, every k-plane must meet X so $\mathcal{C}_X = \mathbb{G}(k,n)$.

We won't get to the question posed at the beginning of class today, but I recommend thinking about it before next class. Here's something to consider.

For $S \subset \mathbb{P}^3$ such a surface,

$$\{L \in \mathbb{G}(1,3) : L \subset S\} \subset \mathbb{G}(1,3)$$

is a closed subvariety. More generally, for any $X \subset \mathbb{P}^n$,

$$\{L \in \mathbb{G}(1,n) : L \subset X\} \subset \mathbb{G}(1,n)$$

is a closed subvariety.

Also, the answer to the problem depends on d.

25. 3/25/16

25.1. **Review.** Recall our problem from last time.

Question 25.1. Let $S \subset \mathbb{P}^3$ be a general surface of degree d. That is, $S = V(f)$ for f a general homogeneous polynomial of degree d. Does S contain a line?

To answer this question, we invoke the following theorem, without proof. We will come back and prove it later.

Theorem 25.2. *Suppose X and Y are irreducible projective varieties and $f : X \to Y$ a regular map. For $q \in Y$, let $\lambda(q) = \dim f^{-1}(q)$. Then, λ is an upper-semicontinuous function. And, if $\lambda_0 = \min_q \lambda(q)$, then $\dim X = \dim Y + \lambda$.*

Last time, we used this to show

$$\dim G(k,n) = k(n-k).$$

Remark 25.3. To prove this, we used the Grassmannian is irreducible. This holds because we have a surjection

$$\mathrm{PGL}_n \to G(k,n).$$

Here, PGL_n is all $n \times n$ invertible matrices, modulo scalars. This is a subset of $\mathbb{P}^{n^2-1}$ which is the complement of the degree n hypersurface which is the determinant of the entries. If we fix a k dimensional subspace $\Lambda_0 \in \mathbb{k}^n$. Then, our quotient map sends $A \mapsto A(\Lambda_0)$. Since this acts transitively, the Grassmannian is irreducible.

Exercise 25.4. Explain the above argument in detail.

Exercise 25.5. Use the above argument and the map from PGL_n to calculate the dimension of $G(k,n)$.

One can also see directly that the Grassmannian is irreducible using the covering of the Grassmannian by charts isomorphic to $\mathbb{A}^{k(n-k)}$.

25.2. **Incidence varieties.**

Definition 25.6. We define the **universal** k**-plane**

$$\Phi = \{(\Lambda, p) \in \mathbb{G}(k,n) \times \mathbb{P}^n : p \in \Lambda\}.$$

We have projections

$$\begin{array}{ccc} & \Phi & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{G}(k,n) & & \mathbb{P}^n. \end{array} \tag{25.1}$$

Lemma 25.7. *The dimension of the universal* k*-plane is*

$$\dim \Phi = (k+1)(n-k) + k = k(n-k) + n.$$

Proof. This follows from Theorem 25.2 using the map π_2 or π_1. □

Lemma 25.8. *Let* $k + l \leq n$*. If* $X \subset \mathbb{P}^n$ *is irreducible of dimension* l *Define* $\mathcal{C}_X = \pi_1(\pi_2^{-1}(X))$*. We can also express*

$$\mathcal{C}_X = \{\Lambda \in \mathbb{G}(k,n) : \Lambda \cap X \neq \emptyset\}.$$

Then, a general k*-plane* Λ *meeting* X *meets* X *in finitely many points.*

Exercise 25.9. Prove this.

Example 25.10. Consider the twisted cubic. Here, $k = l = 1$ and $n = 3$. Then, a general 2-plane meets C. A general line will not meet C, but a general line that does meet C will meet it in only one point.

Definition 25.11. Define the **Fano scheme** of k-planes in X to be

$$F_k(X) := \{\Lambda \in \mathbb{G}(k,n) : \Lambda \subset X\}.$$

Lemma 25.12. *The Fano scheme is a closed subvariety of* $\mathbb{G}(k,n)$*.*

Proof. Consider the map

$$\pi_2^{-1}(X) \to \mathbb{G}(k,n).$$

Here $\pi_2 : \Phi \to \mathbb{P}^n$ is the map from Definition 25.6. Then, the locus of this map $\pi_1|_{\pi_2^{-1}(X)}$ which has fiber dimension k is precisely $F_k(X)$. That is,

$$F_k(X) = \left\{\Lambda \in \pi_2^{-1}(X) : \dim \pi_1(\Lambda) \geq k\right\}.$$

□

25.3. **Answering Question 25.1.** In order to answer Question 25.1, we must first introduce a parameter space parameterizing surfaces of degree d.

Take

$$\mathbb{P}^N = \left\{ \text{homogeneous polynomials of degree } d \text{ in } \mathbb{P}^3 \right\} / \text{ scalars} .$$

Here, $N = \binom{3+d}{3} - 1$. Define

$$\Psi := \left\{ (S, L) \in \mathbb{P}^N \times \mathbb{G}(1,3) : L \subset S \right\}.$$

We have projections

$$\begin{array}{ccc} & \Psi & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ \mathbb{P}^N & & \mathbb{G}(1,3). \end{array} \tag{25.2}$$

We are asking in Question 25.1 whether the map α is dominant (or equivalently, surjective, since β is closed,) or not. Now, the fiber of β over a line $L \in \mathbb{G}(1,3)$ is the set of surfaces containing that line.

Now, we have a restriction map

$$\left\{ \text{homogeneous polynomials of degree } d \text{ on } \mathbb{P}^3 \right\}$$
$$\to \left\{ \text{homogeneous polynomials of degree } d \text{ on } L \cong \mathbb{P}^1 \right\}.$$

The latter is a vector space of dimension $d+1$. Since the map is surjective, the kernel is a subspace of dimension $d+1$. Therefore, the fibers of β have dimension $N-d-1$. That is, the fibers are $\mathbb{P}^{N-d-1}$. Therefore, Ψ is irreducible of dimension $\dim \Psi = 4 + N - d - 1$. So, if $d > 3$, then α cannot be surjective, and so the answer is no when $d > 3$.

We have three remaining cases, when $d = 1, 2, 3$.

(1) When $d = 1$, the varieties are planes, so they contain a two dimensional family of lines.
(2) When $d = 2$, we have a quadric surface, and we know they contain a 1-dimensional family of lines.
(3) When $d = 3$, we might expect that a general cubic surface contains finitely many lines. To prove this, by upper semicontinuity, we only need to show there exists a cubic with finitely many lines. Indeed, the cubic surface $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ has 27 lines which are of the form $x_i = \omega_1 x_j, x_k = \omega_2 x_l$ where i, j, k, l are distinct indices and ω_1, ω_2 are cube roots of unity.

Exercise 25.13. Prove these are the only lines.

(4) If $d \geq 4$, there are no lines.

There are still some more problems to talk about in the case $d \geq 4$. Let's now specialize to $d = 4$, although the questions can be generalized to higher dimensions. When $d = 4$, we've proven a general quartic surface does not contain any lines.

Question 25.14. What do the quartic surfaces which do contain a line look like?

We know such quartic surfaces containing a line are in the image of α. Since $N = 34$ in this case, and $\dim \Psi = 33$. So, if α is generically finite, then the image will be a hypersurface. To prove this is a hypersurface, we would need to show a general quartic surface containing a line contains only finitely many.

Question 25.15. What is the degree of this hypersurface?

The answer is 320, but this is much harder than what we can do in this course.

26. 3/28/16

From before, Joe still owes a proof of the fundamental theorem on dimension. We'll still continue to use it without proof today.

26.1. **Secant Varieties.**

Definition 26.1. A variety $X \subset \mathbb{P}^n$ is **nondegenerate** if there is no hyperplane $H \subset \mathbb{P}^n$ with $X \subset H$.

Let $X \subset \mathbb{P}^n$ be an irreducible projective nondegenerate variety of dimension k. To construct the secant variety, we want to take the union of all secant lines to X.

Definition 26.2. We have a rational map

$$\sigma : X \times X \to \mathbb{G}(1, n)$$
$$(p, q) \mapsto \overline{pq}.$$

The **variety of secant lines to** X is im $\sigma \subset \mathbb{G}(1, n)$. We denote $\mathcal{S}$ or $\mathcal{S}(X)$ to mean im σ. The **secant variety of** X is $\cup_{\ell \in \mathcal{S}} \ell \subset \mathbb{P}^n$.

Fact 26.3. In fact, σ will be regular (and not just birational) if X is a smooth curve, although we haven't defined smooth yet.

Here are some basic questions we can ask about the secant variety.

Question 26.4. What is the dimension of the secant variety to X? Is the secant variety irreducible?

To understand the dimension and irreducibility, given X, we introduce the incidence correspondence

$$\Sigma := \{(\ell, p) \in \mathcal{S}(X) \times \mathbb{P}^n : p \in \ell\} \subset \mathbb{G}(1, n) \times \mathbb{P}^n.$$

We have projections

$$\begin{array}{ccc} & \Sigma & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{S} & & S(X) \end{array} \tag{26.1}$$

Remark 26.5. If the general fiber dimension of σ is positive, then X contains the line joining any two points. In this case, the variety must be a linear subspace of $\mathbb{P}^n$. To see this, think on the vector space level. Containing the line joining any two points in $\mathbb{P}^n = \mathbb{P}V$, means that it contains the plane joining any two lines in V. This means it is a subset of V closed under scaling and addition. Hence, it is a linear subspace of V, and hence a linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$.

26.1.1. *Irreducibility of* $S(X)$. First, note that X is irreducible, so $X \times X$ is as well. The image of an irreducible variety is irreducible, so $\mathcal{S}$ is irreducible. The fibers of π_1 are all 1 dimensional, so Σ is irreducible. This implies that $S(X)$ is irreducible, because the image of an irreducible variety is irreducible. This answers the irreducibility question.

26.1.2. *Dimension of* $S(X)$. We now use Σ to examine the dimension of $S(X)$. Recall that $\mathcal{S}$ has dimension $2k$, and so Σ is irreducible of dimension $2k+1$. So, we "expect" that the dimension of $S(X)$ is $\min(n, 2k+1)$ (the latter in the case that π_2 is generically finite).

26.2. **Deficient Varieties.**

Definition 26.6. We say X is deficient if

$$\dim S(X) < \min(n, 2k+1).$$

Fact 26.7. No nondegenerate curve is deficient.

Fact 26.8. There exists a unique deficient surface, which is the 2-Veronese.

Example 26.9. We will show that the 2-Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is deficient. Let's notate it as X.

We'll give two solutions.

26.2.1. *The Veronese surface is deficient: Proof 1.* Recall this is given by

$$\begin{aligned} \nu_2 : \mathbb{P}^2 &\to \mathbb{P}^5 \\ [x, y, z] &\mapsto \left[x^2, y^2, z^2, xy, xz, yz\right]. \end{aligned}$$

Observe that a line in $\mathbb{P}^2$ is mapped to a conic in $\mathbb{P}^5$. That is, the image of the line will be contained in a 2 plane in $\mathbb{P}^5$. To see that, when you restrict to a line, there are 6 quadratic polynomials on $L \cong \mathbb{P}^1$, but there are only three independent quadratic polynomials on $\mathbb{P}^1$, so there must be three linear relations. To see this in a simple way, if we take $L := V(z)$, then the map sends

$$\begin{aligned} \nu_2|_L : L &\to \mathbb{P}^5 \\ [x, y] &\mapsto \left[x^2, y^2, 0, xy, 0, 0\right]. \end{aligned}$$

So, the image of a line is a plane conic $C \subset \Lambda \subset \mathbb{P}^5$, with Λ a 2-plane. Suppose $r \in S(X)$ is general. That is, suppose $r \in \overline{\nu_2(p), \nu_2(q)}$ for some $p, q \in \mathbb{P}^2$. Look at the line $L := \overline{pq} \subset \mathbb{P}^2$. Then, $\nu_2(L) = C$, and for any point $r \in \Lambda$ any secant line through r in Λ is a secant line. This implies that there is at least a 1-dimensional family of lines through any point in the secant variety of X. In other words, the fibers of the map

$$\pi_2 : \Sigma \to S(X)$$

are positive dimensional.

Exercise 26.10. Show that, in general, this 1-dimensional family of lines through r are all the lines.

Hence, $S(X)$ has dimension 4, since the fibers are 1 dimensional from a 5 dimensional variety Σ.

26.2.2. *The Veronese surface is deficient: Proof 2.* For simplicity of notation, notate the coordinates on $\mathbb{P}^5$ as $w_0, \ldots, w_5$. We can write down the equations defining the Veronese surface, as we did near the beginning of the course. These equations are

$$\begin{aligned} w_0 w_1 - w_3^2 \\ w_0 w_2 w_4^2 \\ w_1 w_2 - w_5^2 \\ w_3 w_4 - w_0 w_5 \\ w_3 w_5 - w_1 w_4 \\ w_4 w_5 - w_2 w_3 \end{aligned}$$

So, we have

$$X = \left\{ [w] : \operatorname{rk} \begin{pmatrix} w_0 & w_3 & w_4 \\ w_3 & w_1 & w_5 \\ w_4 & w_5 & w_2 \end{pmatrix} \right\} \leq 1$$

That is, the six equations above are precisely the two by two minors of the above matrix.

This also implies that the secant variety is not all of $\mathbb{P}^5$. If we have two points on X, we get two such matrices. The secant variety consists of all linear combinations of two 3×3 matrices of rank 1. So, any linear combination must have rank at most 2. In particular, the secant variety satisfies

$$S(X) \subset \left\{ [w] : \det \begin{pmatrix} w_0 & w_3 & w_4 \\ w_3 & w_1 & w_5 \\ w_4 & w_5 & w_2 \end{pmatrix} \right\}$$

Further, since $S(X)$ is four dimensional (as we saw in proof 1), and the determinant is irreducible. Therefore, this containment is an equality, and $S(X)$ is a cubic hypersurface.

Remark 26.11. Let's explain how to show that the determinant of

$$\begin{pmatrix} w_0 & w_3 & w_4 \\ w_3 & w_1 & w_5 \\ w_4 & w_5 & w_2 \end{pmatrix}$$

as a matrix of linear forms on $\mathbb{P}^5$ is an irreducible cubic hypersurface in $\mathbb{P}^5$.

Let's look at the set of 3×3 matrices of rank at most 2. For now, let's ignore the symmetry conditions, and just look at

$$X := \left\{ A \in \mathbb{P}^8 \cong \mathbb{P}M_{3\times 3} : \operatorname{rk} A \leq 2 \right\}.$$

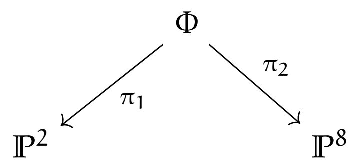
Verifying this condition involves a polynomial of degree 3. The goal is to linearize the problem. To say a 3×3 matrix has rank 2 is equivalent to saying it has a nonzero kernel. If we specified a 1 dimensional subspace in the kernel, we would linearize the problem, since it's then a linear condition as to whether the matrix vanishes on that.

Introduce the incidence correspondence

$$\Phi := \left\{ ([v], [A]) \subset \mathbb{P}^2 \times \mathbb{P}^8 : v \in \ker A \right\}.$$

We have projections

(26.2)



Note that the image of π_2 is precisely X, the locus of rank 2 matrices.

Now, if we fix a vector $v \in \mathbb{P}^2$ the fiber is the set of matrices vanishing on v. It's three linear conditions for a matrix to vanish on a vector in $\mathbb{P}^2$, so the space has codimension 3 in $\mathbb{P}^8$. that is, the fibers are $\mathbb{P}^5 \subset \mathbb{P}^8$. This implies that Φ is irreducible, since it maps to $\mathbb{P}^2$ with irreducible fibers of the same dimension. Then, the image of Φ under π_2 is also irreducible, implying that X is irreducible.

Exercise 26.12. Use a similar method to show the determinant of a symmetric 3×3 matrices in $\mathbb{P}^5$ is irreducible.

We conclude with a question for next time.

Question 26.13. Let

$$M = \{ m \times n \text{ nonzero matrices } / \text{scalars} \} \cong \mathbb{P}^{mn-1}.$$

Let

$$M_k := \{ \text{matrices of rank at most } k \} / \text{scalars} \subset M.$$

What is the dimension of M_k?

27. 3/30/16

27.1. **Schedule.**

(1) Today we'll do more examples of dimension counting.
(2) On Friday, we'll do proofs of the main theorems on dimension.
(3) Next week, we'll start on Hilbert polynomials.

Today, we'll show how we use dimension to prove qualitative theorems about polynomials.

27.2. **The locus of matrices of a given rank.** We will start by defining the locus of rank k matrices inside a projective space of matrices of a fixed size.

Definition 27.1. Let

$$M = \{m \times n \text{ nonzero matrices }\} / \text{ scalars }.$$

Note that $M \cong\cong \mathbb{P}(\hom(V, W)) \cong \mathbb{P}^{mn-1}$, where V is m-dimensional and W is n-dimensional.

Then, for $1 \leq k < \min(m, n)$, we can define

$$M_k := \{\phi \in \hom(V, W) : \operatorname{rk} \phi \leq k\}$$

which is a closed subvariety of M.

A natural question to ask is the following:

Question 27.2. What is $\dim M_k$? Is M_k irreducible?

Let's start by answering this in some special cases.

(1) When $k = 1$, then the matrix has a 1 dimensional image and a $k-1$ dimensional kernel. Such a matrix is completely specified, up to scalars. Therefore,

$$\{\text{ matrices of rank } 1\} / \text{ scalars } \cong \mathbb{P}V^\vee \times \mathbb{P}W \cong \mathbb{P}^m \times \mathbb{P}^n.$$

So, in this case, it is irreducible of dimension $m + n - 2$.

(2) If $m = n$ and $k = m - 1$, this is just the vanishing locus of the determinant polynomial. This is irreducible of dimension $n^2 - 2$, which we proved last time.

Proposition 27.3. *The space* M_k *is irreducible and* $\dim M_k = (mn - 1) - (m-k)(n-k)$.

Remark 27.4 (Idea of proof of Proposition 27.3)**.** We will now compute the dimension and show M_k is irreducible using the technique of linearization.

We're pretty good at linear algebra, but we kind of suck at problems involving higher degree polynomials. So, if possible, one often wants to reduce high degree polynomials to linear polynomials.

The idea is to say that a matrix from an m-dimensional space to and n dimensional space has rank k is the same as saying that its kernel has dimension $m - k$. So, we'll specify an $m - k$ plane in the source space, and then the set of matrices killing that subspace is a linear subspace of all matrices.

Proof. Introduce the incidence correspondence

$$\Phi := \{(\Lambda, \phi) \in G(m-k, V) \times M : \phi(\Lambda) = 0\}.$$

We have projections

(27.1)
$$\begin{array}{ccc} & \Phi & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ G(m-k, V) & & M \end{array}$$

Observe that, by construction im $\pi_2(\Phi) = M_k \subset M$. Now, the fibers of π_1 are just linear subspaces of M, because the locus of matrices with a fixed kernel is a linear subspace.

$$\dim G(m-k, m) = m(m-k).$$

Also, the fibers of π_1 are all $nk-1$ dimensional, because they are isomorphic to

$$\mathbb{P}(\hom(V/\Lambda, W)) \cong \mathbb{P}^{nk-1}$$

Therefore, Φ is irreducible of dimension $k(m-k)+nk-1$.

In particular, im $\pi_2(\Phi) = M_k$ is irreducible. It only remains to calculate $\dim M_k$. If $T \in M_k$ is a matrix of rank precisely k, then $\pi_2^{-1}(T)$ has a single point in its fiber. Note that the preimage of a matrix of rank less than k will be more than zero dimensional. But, the general fiber is 0 dimensional, since the locus of matrices of rank exactly k is open in the locus of matrices with rank at most k. Therefore,

$$\begin{aligned} \dim M_k &= k(m-k) + nk - 1 \\ &= k(m+n-k) - 1 \\ &= (mn-1) - (m-k)(n-k). \end{aligned}$$

In particular, $M_k \subset M$ is irreducible of codimension $(m-k)(n-k)$. □

Remark 27.5. There was a graduate student here around 20 years. He was a bright guy, but he was addicted to proof by contradiction. When you set up a proof by contradiction, you set up a proof where it's in your interest to make a mistake. And this is dangerous. For example, if you want to prove the Riemann hypothesis, you assume the negation of the Riemann hypothesis. You then do a long calculation into which you insert a tiny mistake, and then you deduce the Riemann hypothesis. This grad student, when he made a mistake, he figured that one of his hypotheses was wrong.

In particular, this grad student was dealing with 2×3 matrices. He wanted to know what the equations defining the rank 1 locus. Well, there are three quadratic equations which are the two by two minors. So, this student's calculation said that the locus of rank 1 matrices should have codimension 3, because an intersection of three quadrics has codimension 3. (Of course, this is completely bogus, but the calculation was consistent.)

In fact, the grad student, after doing a very long calculation, had Mumford in the audience. Mumford was reading his mail, mostly bored. Eventually, Mumford looked up, and said "that 3 should be a 2." Then, the grad student was flabbergasted and embarrassed.

Remark 27.6. Another interesting case is when we are looking at rank 1 2×3 matrices. The vanishing locus of these turn out to be twisted cubics.

27.3. **Polynomials as determinants.** In this subsection, we consider the following question.

Question 27.7. When is a general homogeneous polynomial of degree d in n variables expressible as the determinant of a $d \times d$ matrix of linear forms.

Example 27.8. Say $n = 3$. Let's look at

$$N_{3,d} := \{d \times d \text{ matrices of linear forms in } x, y, z\} \, / \text{ scalars} .$$

This is a projective space of dimension $3d^2 - 1$, since each entry is three dimensional, as there are 3 variables, and there are d^2 entries. We have a rational map

$$\pi : N_{3,d} \dashrightarrow \mathbb{P}^{\binom{d+2}{2}-1}.$$

This sends a matrix to a polynomial of degree d. This will only be a rational map, since if the matrix has determinant which is identically 0, the map will not be defined. This will happen when, for instance, the matrix has a row of all 0's.

To find whether this is a dominant map, it suffices to find the dimension of the fiber of π. If we have a given matrix of linear forms (L_{ij}), if we multiply on the left or right by any invertible matrix of scalars, we get another matrix with the same determinant, up to scalars.

Exercise 27.9. Show that, in fact, conversely, any matrix with the same determinant is related by multiplication on the left and right by an invertible scalar matrix.

So, the fibers of π are isomorphic to $PGL_d \times PGL_d$, where PGL_d is the (d^2-1)-dimensional projective space of invertible $d \times d$ matrices modulo scalars. Therefore, the dimension of the general fiber of π is $2(d^2-1)$. So, we expect a polynomial of degree d in x, y, z is expressible as a determinant if and only if

$$3d^2 - 1 - 2(d^2-1) \geq \binom{d+2}{2} - 1.$$

Or, simplifying, we want

$$d^2 + 1 \geq \binom{d+2}{2} - 1.$$

It is simple to see this is satisfied. Making a table, we have

d	$\binom{d+2}{2} - 1$	d^2+1
2	5	5
3	9	10
4	14	17

TABLE 1. A table comparing the expected dimension and actual dimension of the space of polynomials expressible as a determinant

Let's make the analogous table for $n = 4$. In this case, we can set up the analogous map, in which case we want to know

$$4d^2 - 1 - 2(d^2-1) \geq \binom{d+3}{3} - 1.$$

We can make a table of these charts to see these are satisfied.

d	$\binom{d+3}{3} - 1$	$2d^2+1$
2	9	9
3	19	19
4	34	33

TABLE 2.

Here is a challenge question:

Exercise 27.10. Suppose $S \subset \mathbb{P}^3$ is a quartic surface. Show that S is determinantal if and only if S contains a twisted cubic curve.

28. 4/1/16

28.1. **Proving the main theorem of dimension theory.** Recall the main theorem we've been using.

Theorem 28.1. *Let* X *and* Y *be irreducible projection varieties and let* $f : X \to Y$ *be a surjective map. For* $q \in Y$*, set*

$$\lambda(q) := \dim f^{-1}(q).$$

Then, λ *is upper semicontinuous. Further, if* $\lambda_0 = \min_{q \in Y} \lambda(q)$*, then*

$$\dim X = \dim Y + \lambda_0.$$

Today, we'll prove this. In order to do so, we introduce the following local version of the theorem.

Theorem 28.2. *Suppose* X *and* Y *are irreducible projective varieties and let* $f : X \to Y$ *be any regular map. For* $p \in X$*, set*

$$\mu(p) := \dim_p f^{-1}(f(p)).$$

Then, μ *is upper semicontinuous. Let* $X_0 \subset X$ *be an irreducible component. If* $\mu_0 = \min_{p \in X_0} \mu(p)$ *and* Y_0 *is the closure of* $f(X_0) \subset Y$*, then*

$$\dim X_0 = \mu_0 + \dim Y_0.$$

Proof of Theorem 28.1 assuming Theorem 28.2. We have to show that

$$\{q \in Y : \lambda(q) = m\} \subset Y$$

is closed. But, we can write

$$\lambda(q) = \max_{p \in f^{-1}(q)} \mu(p),$$

and so

$$\{q \in Y : \lambda(q) = m\} = f\left(\{p \in X : \mu(p) \geq m\}\right)$$

But, $\{p \in X : \mu(p) \geq m\}$ is a closed set, by Theorem 28.2. Hence, the image is also closed because f is a closed map (as any map of projective varieties is closed). □

It only remains to prove the local version.

Proof of Theorem 28.2. We can restrict to the case that X and Y are both affine, since the theorem is local on the source and target of f. So, let $X \subset \mathbb{A}^m, Y \subset \mathbb{A}^n$, and write

$$\begin{aligned} f : X &\to Y \\ (x_1, \ldots, x_m) &\mapsto \left(f_1(x), \ldots, f_n(x)\right). \end{aligned}$$

We have $f_\alpha \in A(X)$, where $A(X)$ is the homogeneous coordinate ring of X. No, re-embed $X \hookrightarrow \mathbb{A}^{m+n}$ by the map

$$\begin{aligned} X &\to \mathbb{A}^{m+n} \\ (x_1, \ldots, x_m) &\mapsto (f_1(x), \ldots, f_n(x), x_1, \ldots, x_m). \end{aligned}$$

So, the map $f : X \to Y$ factors into a series of projections
(28.1)

$$\mathbb{A}^{m+n} \longrightarrow \mathbb{A}^{m+n-1} \longrightarrow \cdots \longrightarrow \mathbb{A}^{n+1} \longrightarrow \mathbb{A}^n.$$

$$X := X_m \longrightarrow X_{m-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow Y.$$

Here, each map is

$$\begin{aligned} \mathbb{A}^l &\to \mathbb{A}^{l-1} \\ (x_1, \ldots, x_l) &\mapsto (x_1, \ldots, x_{l-1}). \end{aligned}$$

Therefore, it suffices to prove the theorem separately for each such intermediate map. That is, it suffices to prove Lemma 28.3.

Lemma 28.3. *We have that Theorem 28.2 holds for the map π of the form*

(28.2)
$$\begin{array}{ccc} \mathbb{A}^l & \xrightarrow{\pi} & \mathbb{A}^{l-1} \\ \uparrow & & \uparrow \\ X & \longrightarrow & Y \end{array}$$

Proof. For such a map π, the corresponding function $\mu(p)$ is given by

$$\mu(p) = \begin{cases} 1 & \text{if } \pi^{-1}(\pi(p)) \subset X \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to prove that the locus where $\mu(p) = 1$ is a closed subset of X.

Say

$$X := V(g_\alpha)$$

where we write the polynomials defining X as

$$g_\alpha(z_1, \ldots, z_l) := \sum a_{\alpha,i}(z_1, \ldots, z_{l-1}) z_l^i$$

In other words, we are writing the functions defining X as a polynomial in the last variable with coefficients in the first $l-1$ variables. Then, the locus on which $\mu(p) = 1$ is precisely $V(a_{\alpha,i})$. □

□

28.2. **Fun with dimension counts.** Recall the following question from previous lectures.

Question 28.4. Does a general surface $S \subset \mathbb{P}^3$ of degree d contain any lines?

To prove this, we considered the parameter space of these surfaces given by homogeneous polynomials of degree d, which was a projective space $\mathbb{P}^N$, where $N = \binom{d+3}{3} - 1$. Then, we introduced the incidence correspondence

$$\Phi := \left\{ (S, L) \in \mathbb{P}^N \times \mathbb{G}(1,3) \right\}$$

which had projections

$$\begin{array}{ccc} & \Phi & \\ \swarrow & & \searrow \\ \mathbb{P}^N & & \mathbb{G}(1,3). \end{array} \tag{28.3}$$

We noted that the fibers of the left map were $\mathbb{P}^{N-d-1}$, and so Φ is irreducible of dimension $N - d + 3$, ad we concluded that the left projection map could not be dominant when $d > 3$.

Remark 28.5. The two key ingredients were having a parameter space for lines, which was the grassmannian, and a parameter space of surfaces, which was a projective space.

It was key that we had the parameter spaces, and then the result followed from a not too difficult dimension count.

Here is another question along the same lines.

Question 28.6. Does a general surface $S \subset \mathbb{P}^3$ of degree d contain any twisted cubics?

We have a parameter space for surfaces of degree d. However, we don't have a parameter space for twisted cubics. This makes it difficult to answer this question in an analogous fashion to Question 28.4. In other words, we need a variety $\mathcal{H}$ so that the points of $\mathcal{H}$ correspond bijectively to twisted cubic curves, and further we have a projective variety

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{H} \times \mathbb{P}^3 \\ \downarrow & \swarrow & \\ \mathcal{H} & & \end{array} \tag{28.4}$$

with a map $\mathcal{C} \to \mathcal{H}$ so that the fiber over a point $[C] \in \mathcal{H}$ is the curve C. (This variety $\mathcal{H}$ is called a Hilbert scheme and $\mathcal{C}$ is called a universal family over the Hilbert scheme.)

Remark 28.7. For the remainder of today, we will just assume the existence of these parameter spaces and use this to do dimension counts.

Question 28.8. What is the dimension of the space of twisted cubics?

Lemma 28.9. *The parameter space for twisted cubics is irreducible and* 12 *dimensional.*

We give two "proofs," or really, two explanations since we haven't defined the parameter space.

Proof 1 of Lemma 28.9. To answer this question, recall that a twisted cubic curve is defined to be a curve projectively equivalent to the map

$$\begin{aligned} f : \mathbb{P}^1 &\to \mathbb{P}^3 \\ [x_0, x_1] &\mapsto \left[x_0^3, x_0^2 x, x_0 x_1^2, x_1^3\right]. \end{aligned}$$

Define $C_0 := \operatorname{im} f$. We then have a map

$$\begin{aligned} \phi : \mathrm{PGL}_4 &\to \mathcal{H} \\ A &\mapsto A(C). \end{aligned}$$

where $\mathrm{PGL}_4 \subset \mathbb{P}^{15}$ is the nonsingular matrices, modulo scalars. However, ϕ has positive dimensional fibers. If we applied an automorphism of $\mathbb{P}^1$, and then applied f, the image would agree, but we would have different maps.

Exercise 28.10. Show that two twisted cubics agree if and only if they are related by precomposing with an automorphism of $\mathbb{P}^1$.

So, in fact the fibers of ϕ are isomorphic to $\mathrm{PGL}_2 \subset \mathbb{P}^3$. This implies that the map ϕ is from an irreducible 15 dimensional variety to another irreducible variety with 3 dimensional fibers. This implies $\dim \mathcal{H} = 12$. □

Proof 2 of Lemma 28.9. Now, it's not to difficult to show the following result, which is proven in the first chapter of our textbook "A first course."

Theorem 28.11. *If* $p_1, \ldots, p_6 \in \mathbb{P}^3$ *are any 6 points, which no four coplanar, then there exists a unique twisted cubic passing through* $p_1, \ldots, p_6$.

So, we have an open subset $U \subset (\mathbb{P}^3)^6$, where

$$U := \{(p_1, \ldots, p_6) : \text{ no four points are coplanar } \}.$$

We then have an incidence correspondence

$$\Psi := \{(p_1, \ldots, p_6, C) \in U \times \mathcal{H} : p_1, \ldots, p_6 \in C\}.$$

We have projections

$$\begin{array}{ccc} & \Psi & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{H} & & (\mathbb{P}^3)^6. \end{array} \tag{28.5}$$

From Theorem 28.11, we have that π_2 has zero dimensional fibers, and so we obtain that $\dim \Psi = \dim (\mathbb{P}^3)^6 = 18$. We then have that π_1 has fibers which are isomorphic to an open subset of C^6, and hence are 6 dimensional. Therefore, $\dim \mathcal{H} + 6 = \dim \Psi = 18$, and so $\dim \mathcal{H} = 12$. □

Given $C = \phi(\mathbb{P}^1) \subset \mathbb{P}^3$, we have the pullback map

$$\begin{array}{c} \left\{ \text{ polynomials of degree } d \text{ on } \mathbb{P}^3 \right\} \\ \Big\downarrow \alpha \\ \left\{ \text{polynomials of degree } 3d \text{ on } \mathbb{P}^1 \right\}. \end{array} \tag{28.6}$$

This is a map from an $N+1$ dimensional vector space to a $3d+1$ dimensional vector space. We have an incidence correspondence

$$\Phi := \left\{ (S, C) \in \mathbb{P}^N \times \mathcal{H} : C \subset S \right\}.$$

We have projections

$$\begin{array}{ccc} & \Phi & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}^N & & \mathcal{H} \end{array} \tag{28.7}$$

We found from Lemma 28.9 that $\dim \mathcal{H} = 12$. Further, the fibers of π_2 are isomorphic to $\mathbb{P}^{N-3d-1}$, and so Φ is irreducible of dimension $N - 3d - 1 + 12$ dimensional. In particular, the map π_1 cannot be dominant when d is large enough. (Say, bigger than 5.)

29. 4/4/16

29.1. **Overview.** Today, we'll talk about Hilbert functions and Hilbert polynomials

On Wednesday, we'll move onto tangent spaces, which is chapter 14

29.2. **Hilbert functions.** We begin by motivating the study of Hilbert functions. One of the questions we ask about any variety is:

Question 29.1. What sort of polynomials vanish on the variety?

To phrase this question more precisely, define the graded ring

$$S := S(\mathbb{P}^n) = \mathbb{k}[z_0, \ldots, z_n].$$

We grade S by

$$S = \oplus_m S_m.$$

so that S_m is degree m polynomials in $n+1$ variables. For X a variety, $I(X)$ is the set of homogeneous polynomials in S vanishing on X, and we define

$$I(X)_m := I(X) \cap S_m.$$

so that

$$I = \oplus_m I(X)_m.$$

Recall that

$$A(X) = S/I(X) = \oplus_m S_m/I(X)_m.$$

We write $A(X)_m := S_m/I(X)_m$. is the homogeneous coordinate ring of X. In other words, given a projective variety $X \subset \mathbb{P}^n$, we want to know,

Question 29.2. What is the dimension of the vector space of homogeneous polynomials of degree m vanishing on X? That is, what is $\dim I(X)_m$?

This is how we define the Hilbert function.

Definition 29.3. Let $X \subset \mathbb{P}^n$ be a projective variety. The Hilbert function is a map

$$h_X : \mathbb{Z} \to \mathbb{Z}$$

defined by

$$h_X(m) = \dim A(X)_m.$$

Remark 29.4. From the definitions, we have

$$\begin{aligned} h_X(m) &= \dim A(X)_m) \\ &= \operatorname{codim}(I(X)_m \subset S_m) \\ &= \binom{m+n}{n} - \dim I(X)_m, \end{aligned}$$

where here dimension means dimension of vector spaces.

Example 29.5. Suppose $X = \{p_1, p_2, p_3\} \in \mathbb{P}^2$ is three distinct points. We have To calculate $h_X(2)$, we have an map

m	$h_X(m)$
0	1
1	3 if p_i are collinear and 2 otherwise
2	3
3	3

TABLE 3. Table of the Hilbert polynomial of three points in $\mathbb{P}^2$

(29.1) $$I(X)_2 \longrightarrow S(\mathbb{P}^2)_2 \xrightarrow{(ev_{p_1}, ev_{p_2}, ev_{p_3})} \mathbb{k}^3.$$

In fact, this sequence is exact.

Exercise 29.6. Show the sequence is exact, except possibly that the last map is not surjective.

We claim that the evaluation map is surjective. To show this, we only need show we can find a quadratic polynomial with $Q(p_i) = Q(p_j) = 0$ but $Q(p_k) \neq 0$. The same procedure is possible for polynomials of degree at least 2. Hence, in general, $h_X(m) = 3$,

Example 29.7. In general, suppose $X = \{p_1, \ldots, p_d\} \subset \mathbb{P}^n$ is a collection of d distinct points. Then,

$$h_X(m) = d$$

whenever $m \geq d-1$.

To see this, consider the evaluation map

$$I(X)_m \longrightarrow S(\mathbb{P}^n)_m \xrightarrow{\text{ev}} \Bbbk^d \tag{29.2}$$

where ev is the evaluation map at all points. This is surjective whenever $m \geq d-1$.

That is, for all k, there exists F of degree m so that

$$F(p_i) = 0$$

for $i \neq k$ and

$$F(p_k) \neq 0.$$

That is, the values of $h_X(m)$ will vary up to $m = d$, depending on the location of the points. But, after a certain point (namely once $m \geq d$, this Hilbert function becomes constant.

Remark 29.8. If the points $p_1, \ldots, p_d$ are general, (meaning that the lie in a particular open dense subset of $(\mathbb{P}^n)^d$) from a homework problem, we say that the map

$$S(\mathbb{P}^n)_m \to k^d$$

has maximal rank. That is, the rank of the map is $\min(\dim S(\mathbb{P}^n)_m) \to k^d$. So, when m is small relative to d, the map is injective, and it is surjective when m is large relative to d. Hence, for general points,

$$h_X(m) = \begin{cases} \binom{m+n}{n} & \text{if } \binom{m+n}{n} \leq d \\ d & \text{otherwise} \end{cases}$$

Exercise 29.9. The Hilbert function is always increasing. *Hint:* Show that the rank of the map

$$S(\mathbb{P}^n)_m \to k^d$$

is increasing. For this, use the idea of multiplying by a linear polynomial to go between degrees m and $m+1$.

Example 29.10. Let's find the Hilbert function of twisted cubics, although it's not hard to generalize this to all Veronese varieties.

A twisted cubic is the image of the Veronese map

$$\begin{aligned} \phi : \mathbb{P}^1 &\to \mathbb{P}^3 \\ [x, y] &\mapsto [F_0, \ldots, F_3], \end{aligned}$$

where $F_0, \ldots, F_3$ span the space of cubic polynomials on $\mathbb{P}^1$. For example, we can take $F_i = x_0^i x_1^{3-i}$.

We know there are no linear polynomials vanishing on X, because a twisted cubic is not contained in a plane. We know there is a three dimensional vector space of homogeneous degree 2 polynomials vanishing on X, and in fact, these quadrics generate the ideal of X.

We have a map

$$S(\mathbb{P}^3)_m \to S(\mathbb{P}^1)_{3m}.$$

We claim that this map is surjective. This holds because we can write any degree $3m$ polynomial on $\mathbb{P}^1$ as a product of m degree 3 monomials. Hence, the kernel of this map has equal to the codimension of $S(\mathbb{P}^1)_{3m} \subset S(\mathbb{P}^3)_m$.

Exercise 29.11. Show this codimension is $3m+1$.

This shown that the Hilbert polynomial is $3m+1$ when $m \geq 1$.

Example 29.12. Let's now generalize the above example for twisted cubics to arbitrary rational normal curves.

Suppose $X \subset \mathbb{P}^d$ is a rational normal curve given by

$$\begin{aligned} \phi : \mathbb{P}^1 &\to \mathbb{P}^d \\ [x_0, x_1] &\mapsto \left[x_0^d, \ldots, x_1^d\right]. \end{aligned}$$

We have a surjective map,

$$S(\mathbb{P}^d)_m \longrightarrow S(\mathbb{P}^1)_{md} \longrightarrow 0, \tag{29.3}$$

which is part of an exact sequence
(29.4)

$$0 \longrightarrow I(\phi(\mathbb{P}^1)) \longrightarrow S(\mathbb{P}^N)_m \longrightarrow S(\mathbb{P}^1)_{md} \longrightarrow 0$$

which shows $h_X(m) = dm + 1$.

Example 29.13. Let's further generalize the above example to arbitrary Veronese varieties. Recall a Veronese variety is given by the image of the map

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^N,$$

where $N = \binom{n+d}{n} - 1$, which is given by homogeneous polynomials of degree d on $\mathbb{P}^n$. We again have an exact sequence
(29.5)

$$0 \longrightarrow I(\nu_d(\mathbb{P}^n)) \longrightarrow S(\mathbb{P}^N)_m \longrightarrow S(\mathbb{P}^n)_{md} \longrightarrow 0,$$

so we have

$$h_X(m) = \binom{md+n}{n}.$$

Example 29.14. Suppose $X = V(f) \subset \mathbb{P}^2$, for f a homogeneous polynomial of degree d, possibly reducible, but with no repeated factors.

Consider the sequence

$$0 \longrightarrow I(X)_m \longrightarrow S(\mathbb{P}^2)_m \longrightarrow A(X)_m \longrightarrow 0 \tag{29.6}$$

What is $\dim I(X)_m$. The ideal is precisely all polynomials divisible by f. So, the ideal is the set of polynomials of the form $f \cdot g$. Therefore,

$$I(X)_m \cong S(\mathbb{P}^2)_{m-d}$$

where the isomorphism from left to right is given by multiplication by f.

Hence,

$$\begin{aligned} h_X(m) &= \binom{m+2}{2} - \binom{m-d+2}{2} \\ &= dm - \frac{d^2-3d}{2}. \end{aligned}$$

Remark 29.15. Observe that very often these Hilbert functions are polynomials, and the leading term is equal to the degree of the polynomial.

Example 29.16. Suppose $X = V(f) \subset \mathbb{P}^n$, and f is a homogeneous polynomial of degree d with no repeated factors. Then, $I(X) = (f)$ lies in an exact sequence

$$0 \longrightarrow I(X)_m \longrightarrow S(\mathbb{P}^n)_m \longrightarrow A(X)_m \longrightarrow 0. \tag{29.7}$$

Here, $I(X)_m \cong S(\mathbb{P}^n)_{m-d}$, where the left map is multiplication by f. We have

$$h_X(m) = \binom{m+n}{n} - \binom{m-d+n}{n}.$$

Again, this agrees eventually with a polynomial, once $m > d$. In fact, the leading term of this degree $n-1$, and the leading term is closely related to d (it's something like $d/(n-1)!$).

30. 4/6/16

30.1. **Overview and review.** Today, we'll finish up Hilbert functions and polynomials. On Friday, we'll move onto tangent spaces.

Let $X \subset \mathbb{P}^n$ be a projective variety. Define

$$\begin{aligned} S &:= S(\mathbb{P}^n) =: \mathbb{k}[z_0, \dots, z_n] \\ I(X) &\subset S(\mathbb{P}^n) \\ A(X) &= S/I(X) \\ h_X(m) &:= \dim A(X)_m = \operatorname{codim}\left(I(X)_m \subset S_m\right). \end{aligned}$$

Here $I(X)$ is the ideal of X, $A(X)$ is the coordinate ring of X, and h_X is the Hilbert function of X. Last time, we examined several examples of this. In fact, we have a surprising theorem of this Hilbert function.

Theorem 30.1. *For any projective variety $X \subset \mathbb{P}^n$, there exists m_0 and a polynomial p_X so that*

$$h_X(m) = p_X(m)$$

for all $m \geq m_0$.

Further, $\dim X = \deg p_X$.

Definition 30.2. The polynomial p_X from Theorem 30.1 is called the Hilbert polynomial of X.

Example 30.3. If $\dim X = 0$ and $X = \{p_1, \dots, p_d\}$. We saw last time that $h_X(m) = d$ for all $m \gg 0$. In fact, this holds for $m \geq d - 1$.

Proof of Theorem 30.1 . The proof is based on the following lemma.

Lemma 30.4. *Let $X \subset \mathbb{P}^n$ be a projective variety. For $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ a general hyperplane, (in the sense that L should not contain an irreducible component of X) we have $Y = H \cap X$ satisfies*

$$h_Y(m) = h_X(m) - h_X(m-1)$$

for $m \gg 0$.

Exercise 30.5. Deduce that Theorem 30.1 holds from Lemma 30.4. *Hint:* Do this by induction on the dimension, using Example 30.3 as the base case and Lemma 30.4 as the inductive step.

Proof of Lemma 30.4. We deduce this from the following two facts.

Lemma 30.6. *Let $H = V(L)$ be a hyperplane, for L a linear form. If H does not contain any irreducible components of X. Then, L is not a zero-divisor in $A(X)$.*

Proof. If L is a zero divisor, there must be some polynomial so that when you multiply it by L the product vanishes on X. If that polynomial M is nonzero, it must be not identically zero on some irreducible component $X_0 \subset X$, since the vanishing locus is closed. If the product $L \cdot M$ is identically 0 on X, then L must vanish on X_0. □

Proposition 30.7. *For* L *a linear form so that* $V(L)$ *does not contain any irreducible component of* X*, we have* $\mathrm{Sat}(I(X), L) = I(X \cap V(L))$.

Remark 30.8. The proof of this is not so hard, but it needs the notion of transversality, which involves defining tangent spaces. We will define tangent spaces in future classes, but we omit the proof of this proposition.

Now, to complete the proof, we claim there is an exact sequence

$$0 \longrightarrow A(X)_{m-1} \xrightarrow{\times L} A(X)_m \longrightarrow A(Y)_m \longrightarrow 0. \tag{30.1}$$

The right map is surjective by definition. The left map is injective because L is not a zero divisor by assumption. Exactness in the middle follows from $m \gg 0$ by Proposition 30.7.

Exercise 30.9. To see this, suppose we have two ideals $I, J \subset S := \Bbbk[z_0, \ldots, z_n]$. If $\mathrm{Sat}(I) = \mathrm{Sat}(J)$, then $I_m = J_m$ for $m \gg 0$. *Hint:* Reduce to the case that $I \subset J$ by replacing J by $I + J$. This will be on the next homework assignment.

□

□

30.2. **An alternate proof of Theorem 30.1, using the Hilbert syzygy theorem.** Recall the definition of binomial coefficients.

Definition 30.10. For $a, b \in \mathbb{Z}$, we define

$$\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!}$$

Remark 30.11. Observe that

$$\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!}$$

is a polynomial in a. If $S = \Bbbk[z_0, \ldots, z_n]$, then

$$\dim S_m = \binom{m+n}{n}$$

when $m \geq -n+1$.

Remark 30.12. Observe that S is a graded ring so $S = \oplus_m S_m$. We have $S_k \cdot S_l \subset S_{k+l}$.

Definition 30.13. By definition, a **graded module** M over a graded ring S is a module of the form $M = \oplus M_l$ satisfying $S_k \cdots M_l \subset M_{l+k}$.

Definition 30.14. A morphism graded modules between graded modules M and N, which are graded modules over the same graded ring S, is a map of modules $\phi : M \to N$ so that $\phi|_{M_l} : M_l \to N_l$. That is, the image of M_l is contained in N_l.

Definition 30.15. Given M a graded module over a graded ring S, we define $M(k)$ to be the graded module defined by

$$M(k)_l = M_{k+l}.$$

That is, $M(k)$ is isomorphic to M as a module (but not necessarily as a graded module), but with degrees shifted.

Example 30.16. If $F \in S$ is a homogeneous polynomial of degree d, then we have a map

$$S(-d) \xrightarrow{\cdot F} S.$$

Theorem 30.17 (Hilbert syzygy theorem)**.** *Say X is a projective variety. Let $I(X) = (F_1, \ldots, F_l)$ with $\deg F_\alpha = d_\alpha$. Then, we have a surjective homomorphism of graded modules*

$$\oplus_\alpha S(-d_\alpha) \longrightarrow I(X) \longrightarrow 0 \tag{30.2}$$

This map will certainly not be an inclusion, because $F_1 1_2 - F_1 \cdot 1_2$ in $\oplus S(-d_\alpha)$ maps to 0 in $I(X)$, where 1_i is the generator of $S(-d_i)$.

Set $M_1 := \ker \phi_0$. This is again finitely generated as a graded module over S, using that S is Noetherian. Suppose we have $g_1, \ldots, g_m$ are generators for M_1, where $g_i \in M_1$ are generators $G_i \in (M_1)_{d_{1,i}}$.

We can extend (30.2) *to an exact sequence*

$$\oplus S(-d_{1,\beta}) \xrightarrow{\phi_1} \oplus S(-d_{0,\alpha}) \xrightarrow{\phi_0} S \longrightarrow A(X). \tag{30.3}$$

Since the map ϕ_1 will not in general be injective, we can take M_2 be $\ker \phi_1$*, and repeat this process.*

The Hilbert syzygy theorem says that this process terminates after at most n steps, where $S = \mathbb{k}[z_0, \ldots, z_n]$.

We will not give a proof of this theorem, as it is beyond the scope of this course. But, we can quickly deduce Theorem 30.1 from Theorem 30.17.

Proof of Theorem 30.1 using Theorem 30.17. From Theorem 30.17, we have an exact sequence
(30.4)

$$0 \longrightarrow \oplus S(-d_{n,r}) \longrightarrow \cdots \longrightarrow \oplus S(-d_{1,\beta}) \xrightarrow{\phi_1} \oplus S(-d_{0,\alpha}) \xrightarrow{\phi_0} S \longrightarrow A(X).$$

Since we have this exact sequence of finite length, if we assume $m \gg 0$ (where m is big enough that $m+n > d_{ij}$ for all i, j, we have;

$$\begin{aligned} h_X(m) &= \dim A(X)_m \\ &= \sum (-1)^i \dim \left(\oplus S(-d_{i,j})\right)_m \\ &= \sum_{i,j} (-1)^i \dim S_{m-d_{ij}} \\ &= \sum (-1)^i \binom{m+n-d_{ij}}{n}. \end{aligned}$$

□

Exercise 30.18. Write out the minimal resolution of a twisted cubic. *Hint:* We know the ideal of a twisted cubic is generated by three quadrics. We're asking to find the relations among the three quadrics. Deduce the resolution for the twisted cubic. Use the ideas in the above proof to check your answer is equal to the Hilbert polynomial we saw in the last class for the twisted cubic.

31. 3/8/16

31.1. **Minimal resolution of the twisted cubic.** We'll start off by giving one example of a minimal resolution: the twisted cubic. The twisted cubic comes to us as the image of

$$\mathbb{P}^1 \to C \subset \mathbb{P}^3$$

given parametrically by

$$[X_0, X_1] \mapsto [X_0^3, X_0^2X_1, X_0X_1^2, X_1^3].$$

If our coordinates on $\mathbb{P}^3$ are $[Z_0, \ldots, Z_3]$, then C is cut out by the 3 quadrics

$$\begin{aligned} Q_1 &= Z_0 Z_2 - Z_1^2 \\ Q_2 &= Z_0 Z_3 - Z_1 Z_2 \\ Q_3 &= Z_1 Z_3 - Z_2^2 \end{aligned}$$

Note that these are the 2×2 minors of the matrix

$$\begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \end{pmatrix}.$$

Exercise 31.1. Verify this. In other words, show that

$$C = \left\{ [Z_0, Z_1, Z_2, Z_3] : \operatorname{rank} \begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \end{pmatrix} \leq 1 \right\}.$$

Let's get our resolution going: we start off with

$$S \to A(C) \to 0$$

where $S = K[Z_0, \ldots, Z_3]$. The kernel of the map onto $A(C)$ is $I(X)$ (by definition). Since we've found generators Q_i for $I(X)$ we can express this as

$$S(-2)^3 \to S \to A(C) \to 0$$

where the first map is $(f_1, f_2, f_3) \mapsto f_1 Q_1 + f_2 Q_2 + f_3 Q_3$.

Remark 31.2. There are potentially many different resolutions, coming from different choices of generators. However, there is an unambiguous "canonical-ish" one, where we take for our generators a basis for the lowest graded pieces needed to generate the ideal. (Its "canonical-ish" because you have to choose a basis.) This is called the minimal resolution.

For the twisted cubic, we know $\dim I(C)_2 = 3$. We knew this because of the pullback

$$\{\text{quadratic polynomials on } \mathbb{P}^3\} \to \{\text{sextic polynomials on } \mathbb{P}^1\}.$$

This map is surjective. The vector space on the left has dimension 10 and the space on the right has dimension 7, so the kernel $I(C)_2$ is 3-dimensional. We can also pull back cubic polynomials:

$$\{\text{cubic polynomials on } \mathbb{P}^3\} \to \{\text{nonic polynomials on } \mathbb{P}^1\},$$

This is a surjective map from a vector space of dimension 20 to a vector space of dimension 10, so its kernel $I(C)_3$ has dimension 10.

Now, if the Q_i generate, then all multiples of them must be in $I(C)_3$. In other words, we have a surjective map

$$I(C)_2 \otimes S_1 \to I(C)_3$$

which is just multiplication. On the left, we have the tensor product of a 3-dimensional vector space with a 4 dimensional vector space, so its dimension 12. Hence, this map has a 2-dimensional kernel, so we must have two relations among the Q_i whose coefficients are linear forms. We could grind it out and find the kernel of this 10×12 matrix representing the map.

On the other hand, here's a trick: consider the matrix

$$\begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_2 & Z_3 \\ Z_0 & Z_1 & Z_2 \end{pmatrix}.$$

Its determinant is zero since it has a repeated row, and if you expand it you get the relation

$$Z_2Q_1 - Z_1Q_2 + Z_0Q_3 = 0.$$

Doing the same thing but repeat the second row gets the other relation,

$$Z_3Q_1 - Z_2Q_2 + Z_1Q_3 = 0.$$

This is a trick that tends to work on determinantal varieties. These are definitely relations in the kernel of the map. In fact, they are independent and they generate the 2-dimensional kernel. Thus, we have maps

$$0 \to S(-3)^2 \to S(-2)^3 \to S \to A(C) \to 0$$

where that first map is given by the matrix

$$\begin{pmatrix} Z_2 & -Z_1 & Z_0 \\ Z_3 & -Z_2 & Z_1 \end{pmatrix}.$$

In summary, we found relations among the relations, and there are no relations among the relations among the relations. Hilbert syzygy theorem is what says this eventually stops.

These resolutions are useful because they provide a satisfying explanation of *why* the Hilbert function is eventually a polynomial. It gives us an exact sequence of graded modules, so if we look just at the pieces of the the same degree, then we have an exact sequence of vector spaces. From this, we can calculate

$$\begin{aligned} h_C(m) = \dim A(C)_m &= \dim S_m - 3\dim S_{m-2} + 2\dim S_{m-3} \\ &= \binom{m+3}{3} - 3\binom{m+1}{3} + 2\binom{m}{3} \\ &= 3m+1 \end{aligned}$$

We're just took the mth graded pieces, setting their alternating sum to zero, and solving for $\dim A(C)_m$. In general, we'd only have equality once we don't get negative numbers in the gradings subscripts. In this example, however, the Hilbert function happens to always equal to the Hilbert polynomial $3m+1$.

Remark 31.3. The Hilbert syzygy theorem, which says this resolution process stops, is an alternative proof that the Hilbert function is a polynomial for large m. However, it requires some hard-core algebra, and we won't prove it in this class. Remember, our first proof of this fact involved saying that the Hilbert function of a hyperplane section is equal to the first difference of the original Hilbert function for large m. Repeatedly taking hyperplane sections then showed that the Hilbert function agrees with a polynomial of degree equal to the dimension for large m.

31.2. **Tangent spaces and smoothness.** A lot of the geometric notions in algebraic geometry that are defined algebraically have their naive notions rooted in simpler categories. As in our discussion of dimension, our job in algebraic geometry is to set up an algebraic analogue of a pre-existing idea. We now want to give a purely algebraic characterization of the tangent space.

We'll start off by thinking about how we define tangent spaces for manifolds. Here, already there's a range of ways of thinking about it. Suppose we have a manifold X and some point $p \in X$. What do we mean by the tangent space? When the theory of manifolds was being developed, a manifold was a subset of $\mathbb{R}^n$ such that X is (locally around p) the zero locus of C^∞ functions $f_\alpha(x_1, \ldots, x_n)$. If we look at the matrix of partials

$$M = \left(\frac{\partial f_\alpha}{\partial x_i}\right),$$

then we define the tangent space of X at p to be

$$T_pX = \ker M \simeq \mathbb{R}^k$$

where $k = \dim X$.

Now we'll come up with an algebraic definition. Remember that in differential geometry, the cotangent space is the functions that vanish at p modulo functions that vanish to order 2. We'll use this idea.

Definition 31.4. The local ring at p

$$\begin{aligned}\mathcal{O}_{X,p} &= \{\text{germs of regular functions at } p\} \\ &= \{(U, f) : U \supset p \text{ open}, f \text{ continuous on } U\}/\sim\end{aligned}$$

where two germs $(U, f) \sim (V, g)$ are equivalent if $f|_{U\cap V} = g|_{U\cap V}$.

Let

$$\mathfrak{m}_p = \{f : f(p) = 0\} \subset \mathcal{O}_{X,p}$$

be the ideal of germs of functions vanishing at p. We have

$$\mathcal{O}_{X,p} \supset \mathfrak{m}_p \supset \mathfrak{m}_p^2.$$

This $\mathfrak{m}_p^2$ is just the square of this ideal. You can think of it as functions vanishing to order at least 2.

Definition 31.5. The Zariski cotangent space to X at p is $\mathfrak{m}_p/\mathfrak{m}_p^2$. The tangent space is $(\mathfrak{m}_p/\mathfrak{m}_p^2)^*$.

Exercise 31.6. Convince yourself that this definition returns your favorite definition of the tangent space in the case that X is a manifold.

Definition 31.7. We say X is smooth at p if $\dim T_pX = \dim X$.

An important property of tangent spaces is that if we have a regular map $f : X \to Y$ sending $p \in X$ to $q \in Y$, then we get a map of tangent spaces. We have a pull back map on the local rings

$$f^* : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}.$$

This map sends $\mathfrak{m}_q \to \mathfrak{m}_p$ and so $\mathfrak{m}_q^2 \to \mathfrak{m}_p^2$ as well. Hence, we obtain a map on the quotient

$$\mathfrak{m}_q/\mathfrak{m}_q^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2.$$

Taking the transpose gives a map in the opposite direction on the duals:

$$(\mathfrak{m}_p/\mathfrak{m}_p^2)^* \to (\mathfrak{m}_q/\mathfrak{m}_q^2)^*$$

We call this map the differential

$$T_pX \xrightarrow{df} T_qY.$$

Next week, we'll look at more examples and some theorems about smoothness. We won't spend as much time on this as the book does. By a week from today, we'll try to start our discussion of degree.

32. 4/11/16

32.1. **Overview.** There are eight classes left! Today, we'll discuss definitions and constructions involving tangent spaces. On Wednesday, we'll discuss related topics. On Friday, we plan to move on to degree. After this, we'll move on to parameter spaces, and discuss the construction of the Hilbert scheme.

32.2. **Definitions of tangent spaces.** Let X be a quasi-projective variety with a given embedding into $\mathbb{P}^n$ over an algebraically closed field $\Bbbk$ of characteristic 0.

Definition 32.1. Let $p \in X \subset \mathbb{P}^n$ be a point in a variety with a given embedding into $\mathbb{P}^n$. Choose an affine neighborhood $p \in \mathbb{A}^n$ and replace X by $X \cap \mathbb{A}^n$ so that $p \in X \subset \mathbb{A}^n$. Suppose $I(X) = (f_1, \ldots, f_k)$. Define

$$M_p := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \end{pmatrix}$$

We define the **extrinsic Zariski tangent space** to X at p in $\mathbb{A}^n$ as $T_pX := \ker M_p$. That is, if we think of $df_\alpha(p)$ as an element of the cotangent space to $\mathbb{A}^n$ at p with

$$T_pX = \operatorname{Ann}\langle df_1(p), \ldots, df_k(p)\rangle.$$

Definition 32.2. Let X be a quasi-projective variety. Then, the **intrinsic Zariski cotangent space** is

$$T_pX^\vee = \mathfrak{m}_p/\mathfrak{m}_p^2$$

Then, the **intrinsic Zariski tangent space** is

$$T_pX = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^\vee$$

Lemma 32.3. *We have* $\dim_p X \le \dim T_pX$.

Proof.

Exercise 32.4. Show this!

□

Definition 32.5. We say X is **smooth at** p if

$$\dim_p X = \dim T_pX.$$

We say X is **singular at** p if

$$\dim_p X < \dim T_pX.$$

Definition 32.6. Define

$$X^{\text{sing}} = \{ \text{ singular points of } X \} \subset X.$$

Lemma 32.7. *We have $X^{sing} \subset X$ is a closed subset and $X^{sm} \subset X$ is an open subset.*

Proof. On the homework. □

Remark 32.8. Say X has an irreducible decomposition $X = \cup_i X_i$. Then,

$$X^{\text{sing}} = \cup_i X_i^{\text{sing}} \cup_{i \neq j} X_i \cap X_j.$$

Example 32.9. If we take the union of three lines $V(xy, xz, yz)$, this is singular only at the intersection of the three lines.

Example 32.10. If we take a parabola $y - x^2$, and rotate this around an axis

Exercise 32.11. The rotation of a variety around an axis is an algebraic variety.

The Zariski tangent space to this variety at the origin is all of $\mathbb{A}^3$, and so the variety is singular at the origin.

Definition 32.12. For $X \subset \mathbb{A}^n$, with

$$I(X) = (f_1, \ldots, f_k),$$

then the **affine tangent space to** X **at** $p = (z_1, \ldots, z_n) \in \mathbb{A}^n$ is defined to be

$$\left\{ (w_1, \ldots, w_n) : \sum_i \frac{\partial f_\alpha}{\partial z_i}(w_i - z_i) = 0 \text{ for all } \alpha \right\}$$

Warning 32.13. The affine tangent space is not the same as the tangent space. Rather, it is an affine translate of the tangent space.

Definition 32.14. Let $X \subset \mathbb{P}^n$ be a variety. Write

$$I(X) = (f_1, \ldots, f_k),$$
$$p = [z_0, \ldots, z_n].$$

The **projective tangent space to** X **at** p, denoted $\mathbb{T}_p X$ is

$$\left\{ [w_0, \ldots, w_n] : \sum \frac{\partial f_\alpha}{\partial z_i}(p) \cdot w_i = 0 \right\}.$$

Note that the definition of projective tangent space looks simpler than that of the affine tangent space, as we do not have extra z_i terms. This simplification follows from the Euler relations.

Lemma 32.15. *Suppose* F *is a homogeneous polynomial of degree* d*. Then*

$$\sum_i \frac{\partial F}{\partial z_i} z_i = d \cdot F.$$

Proof.

Exercise 32.16. Prove this! *Hint:* This is linear so you may assume F is a monomial. Then, check it explicitly for an arbitrary monomial by writing out the derivatives.

□

32.3. **Constructions with the tangent space.** We'll start with the Gauss map.

Definition 32.17. Let $X \subset \mathbb{P}^n$ be a irreducible smooth variety of dimension k. We can define a map

$$\begin{aligned} \mathcal{G} : X &\to \mathbb{G}(k, n) \\ p &\mapsto \mathbb{T}_p X. \end{aligned}$$

This is well defined since X is smooth, so the tangent spaces always have dimension k.

Remark 32.18. If X is singular, there is a Zariski open subset of smooth points. So, we get a rational map, defined on the smooth locus of X

$$\begin{aligned} \mathcal{G} : X &\dashrightarrow \mathbb{G}(k, n) \\ p &\mapsto \mathbb{T}_p X. \end{aligned}$$

This is defined on the smooth locus meaning we get a map

$$\begin{aligned} \mathcal{G} : X^{sm} &\dashrightarrow \mathbb{G}(k, n) \\ p &\mapsto \mathbb{T}_p X. \end{aligned}$$

This may extend to the singular locus, or it may not.

Example 32.19. If we have a curve crossing itself in the plane, the tangent lines will approach one of two different lines. When we apply the Gauss map to such a curve, we will only have a rational map, and it cannot extend over the singular point, because as we approach the point from different directions, we will have different tangent lines.

Definition 32.20. Let X be a projective variety. We define the **locus of tangent k-planes,** to X to be $\tau(X) := \operatorname{im} \mathcal{G} \subset \mathbb{G}(k, n)$.

We can now use this to make another construction known as the tangential variety.

Definition 32.21. Let X be a projective variety. Then, the **tangential variety** is

$$T(X) := \cup_{\Lambda \in \tau(X)} \Lambda \subset \mathbb{P}^n.$$

Lemma 32.22. *The tangential variety to a* k*-dimensional variety* $X \subset \mathbb{P}^n$ *has dimension at most* $2k$ *and is irreducible.*

Proof. Consider the incidence correspondence

$$\Phi := \{(p, q) \in X^{sm} \times \mathbb{P}^n : q \in \mathbb{T}_p X\}.$$

Define $\Psi := \overline{\Phi}$, be the closure. Then, we have maps

$$\begin{array}{ccccc} & & \Psi & & \\ & \swarrow & & \searrow & \\ X & & & & \mathbb{P}^n. \end{array} \tag{32.1}$$

We see that since X is irreducible of dimension k, we have Ψ is irreducible of dimension $2k$, as the fibers are generically k dimensional. Also, Ψ is irreducible, as it is the closure of Φ, which is irreducible as it maps to X^{sm} with irreducible fibers of the same dimension. Therefore, the image second projection is irreducible and has dimension at most $2k$. □

Example 32.23. We have $\dim T(X) \leq 2 \dim X$. If we take a plane, it will be its own tangential variety. In this case, $\dim TX = \dim X$.

Example 32.24. What is the tangential variety to a twisted cubic? We see it will be a surface in $\mathbb{P}^3$. What is the degree of that surface, and what is the polynomial?

Is $T(X)$ smooth or singular?

In this case, it turns out that $T(X)$ is a smooth quadric surface, which one can see because it maps to the twisted cubic where all the fibers are lines, and these lines do not intersect.

Lemma 32.25. *If* X *is a smooth irreducible variety of dimension* k*. Say* $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ *is a hyperplane. Say* H *is tangent to* X *at* $p \in X$*. The locus*

$$\{ \text{tangent hyperplanes to } X\} \subset (\mathbb{P}^n)^\vee$$

is a projective variety.

Proof. We have the incidence correspondence

$$\Phi := \{(p, H) : H \supset \mathbb{T}_p X\} \subset X \times (\mathbb{P}^n)^\vee.$$

We have projections

$$\begin{array}{ccc} & \Phi & \\ \swarrow & & \searrow \\ X & & (\mathbb{P}^n)^\vee. \end{array} \tag{32.2}$$

The left projection has fibers isomorphic to $\mathbb{P}^{n-k-1}$. So the image of the right map, which is the locus of tangent hyperplanes, is the image of a projective variety under a regular map, hence a projective variety. □

33. 4/13/16

33.1. **Overview.** Seven classes left! Today, we'll discuss remaining topics involving notions of smoothness, singularities, and tangent spaces. The main topics, some of which we won't get to, are

(1) Dual varieties (the variety of planes tangent to a variety)
(2) The Lefschetz principle (the equivalence of all algebraically closed fields of characteristic 0, in a certain sense)
(3) Bertini's theorem (smoothness of hyperplane sections of smooth varieties)
(4) Resolution of singularities (finding a smooth variety mapping to a given singular variety)
(5) Nash blow-ups (a blow up procedure replacing a point with its tangent directions)
(6) Every smooth projective variety of dimension k can be embedded in $\mathbb{P}^{2k+1}$
(7) Subadditivity of codimension (a statement about the codimension of the intersection of varieties)

33.2. **Dual Varieties.** Let $X \subset \mathbb{P}^n$ is a projective variety of dimension k. Everything we will say next can also be carried out for reducible varieties by considering one irreducible component at a time.

We have a Gauss map defined by

$$\begin{aligned} \mathcal{G} : X &\to \mathbb{G}(k, n) \\ p &\mapsto \mathbb{T}_p X. \end{aligned}$$

Remark 33.1. Note that X is smooth if and only if this Gauss map is a regular map, since the tangent space will always have the same dimension as the variety, by definition of smoothness.

Definition 33.2. When X is smooth, the dual variety to X, notated X^* is

$$X^\vee := \left\{ H \in (\mathbb{P}^n)^\vee : H \supset \mathbb{T}_p X \text{ for some } p \in X \right\}$$

In general, if X is not smooth, the Gauss map is defined on the smooth locus of X, and the dual variety is the closure of the image of the Gauss map (which will be a rational map).

Lemma 33.3. *If X is a smooth variety, then $X^\vee$ is a variety. Either $X^\vee$ is a hypersurface or a general tangent hyperplane is tangent along a positive dimensional locus in X.*

Proof. First, we handle the case that X is smooth. Set up the incidence correspondence

$$\Phi := \left\{ (p, H) \in X \times (\mathbb{P}^n)^\vee : H \supset \mathbb{T}_p X \right\}.$$

We have maps

$$\begin{array}{ccc} & \Phi & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & & (\mathbb{P}^n)^\vee \end{array} \tag{33.1}$$

and the image π_2 is $X^\vee$.

In the case X is singular, simply take the above incidence correspondence for smooth points of X and hyperplane containing the tangent space, and then take the closure.

Note that the fibers of π_1 are isomorphic to $\mathbb{P}^{n-k-1}$ in $(\mathbb{P}^n)^\vee$. So, the incidence correspondence Φ is always $n - 1$. Hence, either $X^\vee$ is a hypersurface or β has positive dimensional fibers. This means that a general tangent hyperplane is tangent along a positive dimensional locus in X. □

Remark 33.4. There are in fact examples where X is smooth and nondegenerate, but the dual variety is not a hypersurface.

The object Φ from the above proof is sometimes called the contact manifold or the projective conormal bundle.

Proposition 33.5. *Let $X \subset \mathbb{P}^n$ be an irreducible nondegenerate variety (i.e., X is not contained in some hyperplane) then*

$$\left(X^\vee\right)^\vee = X.$$

Remark 33.6. We omit the proof. But, plane curves provide an illustrative example, which we now investigate. If we start with a curve C and a point p, we get a tangent line $\mathbb{T}_p C$. Then, we're claiming that in $C^\vee$, the tangent line to $C^\vee$ at $(\mathbb{T}_p C)^\vee$ is p.

To see this, as a point q on C approaches p, the secant line limits to $\mathbb{T}_p C$. For every point q, we can look at the tangent line to q. This corresponds to a point $(\mathbb{T}_q C)^\vee$ on $C^\vee$. If we look at $r := \mathbb{T}_p C \cap \mathbb{T}_q C$. This point of intersection r is dual to the secant line joining $(\mathbb{T}_q C)^\vee$ and $(\mathbb{T}_p C)^\vee$. This limit r tends to p as q approaches p.

Exercise 33.7. Make the above precise, proving Proposition 33.5 in the case $C \subset \mathbb{P}^2$ is a curve.

Example 33.8. If we start with a variety X that is not a hypersurface, and we take its dual $X^\vee$, if $X^\vee$ is a hypersurface, $X^\vee$ is an example of a variety whose dual is not a hypersurface.

Note that if $X^\vee$ is not a hypersurface, then X itself is an example of a variety whose dual is not a hypersurface.

In sum, either X or $X^\vee$ will be an example of a variety whose dual is not a hypersurface, if X is not a hypersurface.

33.3. **Resolution of Singularities.** In many ways, smooth varieties are easier to work with than singular varieties. Nonetheless, singular varieties arise naturally as intersections, limits, etc.

Question 33.9. If we have a singular variety X, is there a smooth variety $\widetilde{X} \xrightarrow{\pi} X$ birational to it?

The answer is yes in characteristic 0, according to a famous theorem of Hironaka in the 1960s. This is still open in characteristic p. János Kollár has a book on this, which may be the best source.

33.4. **Nash Blow Ups.**

Example 33.10. We have maps $\mathbb{P}^1 \to \mathbb{P}^2$ given by a triple of cubics, so that the image is given by $V(y^2 - x^2(x+1))$. This is a nodal curve. We can also take the map

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^2 \\ [x_0, x_1] &\mapsto \left[x_0^3, x_0^2 x_1 x_1^3\right]. \end{aligned}$$

Question 33.11. If someone just gave us the curve $V(y^2 - x^2(x+1))$, which has a singular point at $x = y = 0$, could we reconstruct the smooth variety $\mathbb{P}^1$ and the map to each of these curves?

This is precisely the question of resolution of singularities in this case.

Note that since these maps are both given by triples of homogeneous polynomials, we can realize these curves as the projection of a twisted cubic from a point in $\mathbb{P}^3$. This holds because the twisted cubic is simply a map given by four cubic polynomials, so after projecting it, the map will be given by a triple of cubic polynomials.

If the point we project from in $\mathbb{P}^3$ lies on a secant line, we will get a nodal curve in $\mathbb{P}^2$, while if we project from a point on a tangent line to the twisted cubic, we get a cuspidal cubic curve.

Remark 33.12. The key to resolution of singularities is blowing up. We can blow up the singular points. We hope that after blowing up enough, we get a smoother variety. Unfortunately the process of blowing up is not algorithmic. For curves, curves only have isolated singular points. However, on higher dimensional varieties, there are more options: you could blow up a point, a curve, a surface, etc. One thing people have been looking for is an algorithmic process for resolving singularities.

Example 33.13. Say we return to the nodal cubic curve $V(y^2 - x^2(x+1))$. In this case, if we approach the node along one branch, we will get one tangent, and if we approach along the other branch, we will get a different tangent.

The point is that the Gauss map

$$\mathcal{G}_C : C \to C^\vee \subset \left(\mathbb{P}^2\right)^\vee$$

is not regular at the singular point. Let Γ denote the graph of $\mathcal{G}_C$ in $C \times \left(\mathbb{P}^2\right)^\vee$. We have projections

$$\begin{array}{ccccc} & & \Gamma & & \\ & \swarrow & & \searrow & \\ C & & & & C^\vee. \end{array} \tag{33.2}$$

This graph does exactly what we want by separating out the two branches of the curve at the singular point. So, the graph of the Gauss map gives a resolution of singularities.

Definition 33.14. Let $X \subset \mathbb{P}^n$ be a k-dimensional variety. We have the Gauss map

$$\mathcal{G} : X \dashrightarrow \mathbb{G}(k, n).$$

Let Γ be the graph of $\mathcal{G}$, which has a map $\Gamma \to X$. This map is called the Nash blow-up of X.

Remark 33.15. The Nash blow up is intrinsic to X, and doesn't depend on how X is embedded in projective space.

Taking the Nash blow up of a singular variety tends to reduce the singularities.

We close today with the following open question.

Question 33.16. Can we achieve a resolution of singularities by iterating the Nash blow-up?

That is, do iterated Nash blow-ups always resolve singularities?

34. 4/15/16

34.1. **Bertini's Theorem.** Today, we'll talk about the degree of a variety. But, before that, we'll talk briefly about one more topic from smoothness, known as Bertini's theorem. Bertini's theorem is quite ubiquitous in algebraic geometry, and is rather simple.

Theorem 34.1 (Bertini's Theorem)**.** *If $X \subset \mathbb{P}^n$ is quasiprojective and smooth, and $H \subset \mathbb{P}^n$ is a general hyperplane, then $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is a general hyperplane, then $X \cap H$ is smooth.*

Before giving a proof, we state some immediate generalizations.

Corollary 34.2. *If $X \subset \mathbb{P}^n$ is quasi projective, and $H \subset \mathbb{P}^n$ is a general hyperplane, then*

$$(X \cap H)_{sing} \subset H \cap X_{sing}.$$

Proof. Replace X by X^{sm}, and apply Theorem 34.1. □

Corollary 34.3. *If $X \subset \mathbb{P}^n$ is smooth and quasi-projective, and $Y \subset \mathbb{P}^n$ is a general hypersurface of degree d, then $X \cap Y$ is smooth.*

Proof. Compose with the d-Veronese map on $\mathbb{P}^n$

$$X \to \mathbb{P}^n \xrightarrow{\nu_d} \mathbb{P}^N,$$

with $N = \binom{n+d}{n} - 1$. The hyperplane sections of $\mathbb{P}^N$ pull back to hypersurface of degree d in $\mathbb{P}^N$. So, a general hyperplane section of X in $\mathbb{P}^N$ is the same as a general degree d hypersurface section in $\mathbb{P}^n$. Then, apply Theorem 34.1 to $X \subset \mathbb{P}^N$. □

We now prove Bertini's theorem.

Proof of Theorem 34.1. Say $X \subset \mathbb{P}^n$ is smooth of dimension k. Let $p \in X$. We claim that $H \cap X$ is singular if and only if $\mathbb{T}_p X \subset H$.

We now set up the incidence correspondence

$$\Phi := \left\{ (p, H) \subset \mathbb{P}^n \times (\mathbb{P}^n)^\vee : p \in X, H \supset \mathbb{T}_p X \right\}.$$

We have projections

$$\begin{array}{ccc} & \Phi & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & & (\mathbb{P}^n)^\vee \end{array} \tag{34.1}$$

The fibers of π_1 are isomorphic to $\mathbb{P}^{n-k-1}$, and so Φ has dimension $n-1$, and the image in $(\mathbb{P}^n)^\vee$ has dimension $n-1$, and so a general hyperplane does not have a singular intersection with X. □

34.2. **The Lefschetz principle.**

Remark 34.4. Bertini's theorem is something that would have been fairly obvious over the complex numbers, $\mathbb{C}$, classically.

It's worth pointing out that over the complex numbers, to say X is smooth means its a submanifold of $\mathbb{P}^n$. We're saying that a general hyperplane section is a manifold. This is an immediate consequence of Sard's theorem. This is the weakest possible form of Sard's theorem.

Theorem 34.5 (Sard's theorem)**.** *If $f : M \to N$ is a differentiable map of C^∞ manifolds then f has a noncritical value. Then f has a non-critical value, there exists $p \in N$ with $f^{-1}(p) \subset M$ a submanifold.*

That is, there is at least one smooth fiber of f.

Proof. Omitted. □

Here is an alternate proof of Bertini's theorem.

Proof of Bertini's theorem, Theorem 34.1 using Sard's theorem, Theorem 34.5. We can then prove Bertini's theorem over the complex numbers from this, applying Sard's theorem to the universal hyperplane section. That is, there is a map from

$$\Omega := \left\{ (H, p) \in (\mathbb{P}^n)^\vee \times \mathbb{P}^n : p \in H \cap X \right\}$$

with maps

$$(34.2)\qquad \begin{array}{ccc} & \Omega & \\ \swarrow & & \searrow \\ (\mathbb{P}^n)^\vee & & X. \end{array}$$

The fibers of the second projection are isomorphic to $\mathbb{P}^{n-1}$. So, Ω is smooth. By Sard's theorem, for a general point in $(\mathbb{P}^n)^\vee$, the fiber is smooth. □

It seems like the above proof of Bertini's theorem using Sard's theorem only works over $\mathbb{C}$. But, there is a general philosophy called the Lefschetz principle. We call it a philosophy, because it is somewhat vague.

Remark 34.6 (Lefschetz principle)**.** Any theorem in algebraic geometry holding over $\mathbb{C}$ holds more generally over an algebraically closed field of characteristic 0.

Question 34.7. In the statement of the Lefschetz principle, what does "any theorem" mean?

an assertion of the existence or nonexistence to the solution of a collection of polynomial equations.

Example 34.8. Let $X = V(f_1, \ldots, f_k)$. If we ask for X to be smooth, we're asking that there is no point on the variety where $f_1, \ldots, f_k$ vanish and the maximal minors of the Jacobian also vanish.

We now use this to deduce Bertini's theorem for arbitrary fields of characteristic 0, using the knowledge that it holds over $\mathbb{C}$.

Proof of Theorem 34.1 over an arbitrary field of characteristic 0, assuming it holds over $\mathbb{C}$. Suppose $\mathbb{k}$ is an arbitrary algebraically closed field of characteristic 0. If a variety X is defined over $\mathbb{k}$, then a variety is defined by finitely many polynomials, which has finitely many coefficients. If we take X defined over $\mathbb{k}$, we can take all the coefficients of these polynomials, and adjoin those coefficients to the rational numbers. That is, X is defined by a finite collection of polynomials f_α with finitely many coefficients $c_{\alpha,I}$. We then have a finite collection of element $\{c_{\alpha,I} \in \mathbb{k}\}$. We can then replace $\mathbb{k}$ by the subfield of $\mathbb{k}$ generated over $\mathbb{Q}$ by $L := \overline{\mathbb{Q}(c_{\alpha,I})}$. Any such field can then be embedded in the complex numbers. We then apply Bertini's theorem over the complex numbers to conclude the intersection with a general hyperplane, and this lets us deduce Bertini's theorem over $\mathbb{k}$. □

Remark 34.9. One of the things we can do over the complex numbers is that we can integrate a vector field to get a collections of integral curves.

If we have a holomorphic vector field, we can integrate it to get a collection of holomorphic arcs. If we start with an algebraic vector field, the integral curves may be transcendental. If we can prove a theorem using integral curves, it may appear to only hold over the complex numbers, but we can use the Lefschetz to port it over to arbitrary algebraically closed fields of characteristic 0.

Remark 34.10. In some senses the first proof of Bertini's theorem is still preferable because it proves Bertini's theorem over arbitrary fields, but the second proof is still a good illustration of the Lefschetz principle.

34.3. **Degree.** We'll now define the notion of degree. If $X \subset \mathbb{P}^n$ is a hypersurface, say $X = V(f)$, we say $\deg X = \deg F$. On the other hand, if X is 0-dimensional if $X = \{p_1, \ldots, p_d\}$, then we say $\deg X = d$.

Remark 34.11. These two notions agree: A 0-dimensional variety is a hypersurface only when we are working in $\mathbb{P}^1$. If we start with an polynomial with no repeated roots, then its degree is exactly equal to the number of solutions, using the fundamental theorem of algebra (that every degree d polynomial has d roots).

We now want to generalize this notion of degree to varieties of arbitrary dimension.

Remark 34.12. Let X be k-dimensional and irreducible. Let Γ be a general $\mathbb{P}^{n-k-2}$ plane, let Λ be a general $\mathbb{P}^{n-k-1}$ plane in $\mathbb{P}^n$, and let Ω be a general $\mathbb{P}^{n-k}$ plane in $\mathbb{P}^n$. If X has dimension k, then $\Lambda \cap X = \emptyset$ but $\Omega \cap X = \{p_1, \ldots, p_d\}$. We have a rational projection map

$$\pi_\Gamma : \mathbb{P}^n \dashrightarrow \mathbb{P}^{k+1}.$$

If we restrict to X, we get a regular map

$$\pi_\Gamma|_X : X \to X_0 \subset \mathbb{P}^{k+1}.$$

Here, X_0 is a hypersurface in $\mathbb{P}^{k+1}$.

Example 34.13. Start with a twisted cubic. If we project from a point Γ we obtain a plane curve, i.e., a hypersurface in the plane. We take Λ to be a line, and when we project the cubic to a line, we get a finite surjective map. If we have a 2-plane Ω, it will meet X in a finite set of points. In this case, it will meet in three points.

Using the above ideas, here are three definitions of degree.

Definition 34.14. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension k.

(1) Let Γ be a general $n-k-2$ plane. Then, we define the **degree** of X to be
$$\deg\left(\pi_\Gamma(X) \subset \mathbb{P}^{k+1}\right)$$
as a hypersurface.

(2) Let Ω be a general $n-k$ plane. Then, $\Omega \cap X$ is a finite collection of points $X \cap \Omega$. We define the **degree** of X to be the number of points in $X \cap \Omega$.

(3) Let $\pi_\Lambda : X \to \mathbb{P}^k$ be the projection map. We get a corresponding map of fraction fields
$$\pi_\Lambda^* : K(\mathbb{P}^k) \to K(X).$$
The **degree** of X is the degree of the field extension $\deg\left(K(X) : K(\mathbb{P}^k)\right)$.

We can now make a more general definition of degree in terms of Hilbert polynomials which agrees with the previous definitions in the case that X is irreducible. We will see this equivalence in Lemma 34.17.

Definition 34.15. Let $X \subset \mathbb{P}^n$ be a projective variety of dimension k. Let the Hilbert polynomial of X be $p_X(m) := a_1 m^k + \cdots$. Here, a_1 is the leading coefficient of the Hilbert polynomial. Then, the **degree** of X is $k! \cdot a_1$.

Exercise 34.16. Show the above definitions of degree in Definition 34.14 agree. *Hint:* For equivalence of the first two, use the idea that a pullback of a hyperplane in $\mathbb{P}^{k+1}$ under π_Γ is a hyperplane. The others equivalences take some more work.

Lemma 34.17. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. The definition of degree from Definition 34.15 agrees with that of Definition 34.14*

Proof. Let X be a k-dimensional variety. Let $H_1, \ldots, H_k$ be k general hyperplanes. Define
$$X_i := X \cap H_1 \cap \cdots \cap H_i.$$
Then, we have
$$p_{X_i}(m) = p_{X_{i-1}}(m) - p_{X_{i-1}}(m-1).$$

Finally $X_k = X \cap \Omega = \{p_1, \ldots, p_d\}$, so $p_{X_k} = d$. This implies

$$p_X(m) = \frac{d}{k^1} m^k + \cdots ,$$

as desired. □

Example 34.18. Recall the twisted cubic is

$$\mathbb{P}^1 \to \mathbb{P}^3$$
$$[x_0, x_1] \mapsto \left[x_0^3, \ldots, x_1^3\right].$$

We saw that the Hilbert polynomial of a twisted cubic is $3m + 1$. So, by Definition 34.15, this has degree 3.

35. 4/18/16

35.1. **Review.** Recall the definition of the degree of an irreducible projective variety.

Definition 35.1. Let $X \subset \mathbb{P}^n$ be an irreducible variety of dimension k. Then, the degree of X is one of the following equivalent integers.

(1) If $\Gamma \cong \mathbb{P}^{n-k-2}$ is a general linear subspace, then define $\pi_\Gamma : X \to \overline{X} \subset \mathbb{P}^{k-1}$, then $\deg X = \deg \overline{X}$ as a hypersurface.
(2) Let $\Lambda \cong \mathbb{P}^{n-k-1}$ be a general linear subspace of $\mathbb{P}^n$. Let $\pi_\Lambda : X \to \mathbb{P}^k$ be the projection from Λ. Then, the degree (which is the size of the fiber of π_Λ over a general point in the case that the base field is of characteristic 0) of the map π_Λ is the degree of X.
(3) Let $\Omega \cong \mathbb{P}^{n-k}$ be a general linear subspace. Then, $\deg X = \#(\Omega \cap X)$.
(4) The degree is equal to the product of $k!$ with the leading coefficient of the Hilbert polynomial p_X.

In general, if X is reducible of dimension k with $X = \cup_i X_i$, the degree of X

$$\deg X := \sum_{X_i : \dim X_i = k} \deg X_i.$$

Remark 35.2. Recall the following result, which is important for making sense of the notion of the degree of the map π_Λ, which was needed to make sense of one of the definitions of degree.

Proposition 35.3. *Let $f : X \to Y$ be a finite surjective map of irreducible projective varieties. Then, there exists an open $U \subset Y$. Then, there is an open set $U \subset Y$ on which $\#f^{-1}(q)$ is constant and equal to* $\deg[K(X) : K(Y)]$.

Proof. Omitted, see the textbook chapter 7 but see remark below. □

Remark 35.4. Recall the idea of the proof, in the case that $Y = \mathbb{A}^k$ and X is affine. If we have a map $X \to Y$ we can factor it by repeatedly projecting from a point. Say $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^l$. So, in total, we will project $n - l$ times. Say $\dim X = k$. The key step is the last projection $X \subset \mathbb{A}^{k+1} \to Y = \mathbb{A}^k$. In this case, X will be a hypersurface. Write $X = V(g)$ with $g = \sum g_\alpha(x_1, \ldots, x_k) x_{k+1}^\alpha$. We can construct a proper closed locus on Y where the number of roots in the last variable is equal to the degree of g. On this open subset away from the roots colliding, the number of preimages is the degree of this map.

35.2. **Bezout's Theorem, part I: A simpler statement.**

Definition 35.5. Let $X, Y \subset \mathbb{P}^n$ be subvarieties of dimension k and l. Let $p \in X \cap Y$. Then, we say X and Y intersect **transversely** at p if X and Y are varieties of dimensions k and l and

$$\dim \mathbb{T}_p X \cap \dim T_p Y = k + l - n.$$

If X and Y intersect transversely, we say $X \cap Y$ is transverse.

Exercise 35.6. Retaining the setup of Definition 35.5, show that if

$$\dim \mathbb{T}_p X \cap \dim T_p Y = k + l - n.$$

then X and Y are both smooth at p.

Definition 35.7. Say that two varieties X and Y intersect **generically transversely** if $X \cap Y$ is transverse at a general point of any irreducible component of $X \cap Y$ and $X \cap Y \neq \emptyset$. If X and Y intersect generically transversely, we say $X \cap Y$ is generically transverse.

Remark 35.8. Note that two varieties X and Y can intersect generically transversely only when $k + l \geq n$, since if $k + l \leq n$, if the intersection were transverse, we would have $X \cap Y = \emptyset$, which is not allowed. In the case that $k + l = n$, $X \cap Y$ is a finite collection of points, and these must all be transverse intersections.

Theorem 35.9 (Bezout's Theorem)**.** *Say $X, Y \subset \mathbb{P}^n$ are irreducible projective varieties of dimensions k, l with $k + l \geq n$. If $X \cap Y$ is generically transverse then*

$$\deg X \cap Y = \deg X \cdot \deg Y.$$

Proof. We'll see the proof in a later class. □

35.3. **Examples of Bezout, part I.**

Example 35.10. Say $X \subset \mathbb{P}^n$ is any k-dimensional subvariety and say $Y = \mathbb{P}^{n-k} \subset \mathbb{P}^n$ is a linear space. Then, Bezout's theorem Theorem 35.9 says that

$$\#(X \cap \mathbb{P}^{n-k}) = \deg X$$

whenever $X \cap \mathbb{P}^{n-k}$ is transverse. In particular, if $C \subset \mathbb{P}^2$ is a curve of degree d, then the intersection of C with a line only fails to be d points only when the line is tangent to C or intersects C at a singular point.

Example 35.11. Take $\nu := \nu_d : \mathbb{P}^n \to X \subset \mathbb{P}^N$ to be the d Veronese variety, with $N := \binom{n+d}{d} - 1$.

Question 35.12. What is $\deg X$?

35.3.1. *Method 1 by pulling back hyperplanes.* To calculate $\deg X$,
Let $\Lambda \cong \mathbb{P}^{N-n} \subset \mathbb{P}^N$ be general. Then,

$$\begin{aligned}\deg X = \#(X \cap \Lambda)& \\ &= \#(X \cap H_1 \cap \cdots \cap H_n) \\ &= \left(\nu^{-1}(H_1) \cap \cdots \cap \nu^{-1}(H_n)\right)\end{aligned}$$

Note that $\nu^{-1}(H_i)$ is a general hypersurface $Z_i \subset \mathbb{P}^n$ of degree d, because the preimage of a general plane in $\mathbb{P}^N$ is a general hypersurface of degree d in $\mathbb{P}^n$. By (a possibly stronger version than we stated in class of) Bertini's theorem, the general hypersurfaces Z_i intersect transversely in $\mathbb{P}^n$. By Bezout's theorem, Theorem 35.9, the intersection of n hypersurfaces of degree d is d^n.

35.3.2. *Method 2 by using Hilbert polynomials.* Consider the map

$$\begin{array}{c}\left\{ \text{homogeneous polynomials of degree } m \text{ in } \mathbb{P}^N \right\} \\ \big\downarrow \\ \left\{ \text{homogeneous polynomials of degree } d \cdot m \text{ in } \mathbb{P}^N \right\}.\end{array} \tag{35.1}$$

Then, we have

$$h_X(m) = \binom{n + dm}{n} = \frac{d^n}{n!} m^n + \cdots,$$

so the degree is d^n.

Example 35.13. Let

$$\nu := \nu_d : \mathbb{P}^n \to X \subset \mathbb{P}^N$$

with $N = \binom{d+n}{n} - 1$. Consider $Z \subset \mathbb{P}^n$ irreducible of dimension k and define $W := \nu(Z)$. That is,

$$\begin{array}{ccc} \mathbb{P}^n & \longrightarrow & \mathbb{P}^N \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W := \nu(Z). \end{array} \tag{35.2}$$

We solve this using two methods, analogously to the previous example.

35.3.3. *Method 1.* Choose hyperplanes $H_i \cong \mathbb{P}^{N-1} \subset \mathbb{P}^N$. We have

$$\begin{aligned} \deg W &= \#\left(W \cap H_1 \cap \cdots \cap H_k\right) \\ &= \#\left(Z \cap \nu^{-1}(H_i) \cap \cdots \cap \nu^{-1}(H_d)\right) \\ &= \deg Z \cdot d^k. \end{aligned}$$

35.3.4. *Method 2.*

Question 35.14. What is the relationship between $p_Z(m)$ and $p_W(m)$?

In fact, we have

$$p_W(m) = p_Z(md),$$

as this is true on the level of Hilbert functions. To see this, note that the codimension of polynomials of degree m on the target is equal to the codimension of polynomials of degree md vanishing on the source. Hence, we have

$$p_Z(md) = \frac{\deg Z}{k!}(md)^k + \cdots,$$

while

$$p_W(m) = \frac{\deg W}{k!} m^k + \cdots.$$

Therefore,

$$\deg W = d^k \cdot \deg Z.$$

36. 4/20/16

36.1. **Overview.** Today, we'll

(1) study degree calculations more,
(2) give a proof of Bezout's theorem,
(3) and state a second version of Bezout's theorem, without proof.

36.2. **More degree calculations.**

Example 36.1. Recall the segre embedding

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(m+1)(n+1)-1}$$
$$([z_0, \ldots, z_n], [w_0, \ldots, w_m]) \mapsto [\ldots, z_i w_j, \ldots].$$

The image of this map is denoted $\Sigma_{n,m}$ and is the Segre variety.

To find the degree, we compute the Hilbert polynomial of this variety.

Question 36.2. What is the Hilbert function of $\Sigma_{m,n}$?

We have a map

(36.1)

$$\{\text{ homogeneous polynomials of degree } k \text{ on } \mathbb{P}^n\}$$
$$\downarrow$$
$$\{\text{ bihomogeneous polynomials of bidegree } (k,k) \text{ on } \mathbb{P}^n \times \mathbb{P}^m\}$$

which is surjective, as every bihomogeneous monomial can be written as a product of pairwise products of monomial.

The kernel of this map is $I(\Sigma_{n,m})_k$, and so the Hilbert function is simply the dimension of the space of bihomogeneous polynomial of bidegree (k,k). That is,

$$\begin{aligned} h_{\Sigma_{n,m}}(k) &= \binom{n+k}{k}\binom{m+k}{k} \\ &= \left(\frac{k^n}{n!} + O(k^{n-1})\right)\left(\frac{k^m}{m!} + O(k^{m-1}\right) \\ &= \frac{k^{m+n}}{n!m!} + O(k^{m+n-1}) \\ &= \binom{m+n}{n}\frac{k^{m+n}}{(m+n)!} + O(k^{m+n-1}) \end{aligned}$$

and so we have $\deg \Sigma = \binom{m+n}{n}$.

In the special case that $m = n = 1$, we know $\Sigma_{1,1}$ is the quadric surface, and indeed

$$\binom{1+1}{1} = 2.$$

Remark 36.3. Many calculation of degree and Hilbert functions can be done by using the cohomology ring of projective space, although we will not be responsible for that material in this class.

We were able to use Bezout's theorem to calculate Hilbert functions of subvarieties of the Veronese variety. The same tactics don't work for subvarieties of Segre varieties, but such calculations can be done using intersection theory, which uses some more advanced machinery.

Example 36.4 (Degree of cones)**.** Say $X \subset \mathbb{P}^n$ is a k dimensional variety. View $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ as a hyperplane.

Now, choose $p \in \mathbb{P}^{n+1} \setminus \mathbb{P}^n$. Let

$$\overline{pX} := \cup_{q \in X} \overline{pq}.$$

Then,

$$\deg \overline{pX} = \# \left(\overline{pX} \cap \Gamma\right)$$

where Γ is a general $n-k$ plane $\mathbb{P}^{n-k} \subset \mathbb{P}^{n+1}$. But, we see

$$\begin{aligned} \deg \overline{pX} &= \# \left(\overline{pX} \cap \Gamma\right) \\ &= \#(X \cap \pi_p \Gamma) \\ &= \deg X \end{aligned}$$

Here, we are using that $\pi_p(\overline{pX} \cap \Gamma) = X \cap \pi_p \Gamma$, for a general $(n-k)$-plane Γ. So, the degree of a cone over X is equal to the degree of X.

Example 36.5 (Degree of projections)**.** Suppose $X \subset \mathbb{P}^{n+1}$ is a k dimensional variety. Let $p \notin X$. We will find $\deg \pi_p(X) \subset \mathbb{P}^n$. Let Λ be a general $(n-k)$-plane. Then, if the projection map π is generically one to one,

$$\begin{aligned} \deg\left(\pi_p(X)\right) &= \#\left(\Lambda \cap \pi_p X\right) \\ &= \#\left(\overline{p\Lambda} \cap X\right) \\ &= \deg X. \end{aligned}$$

where we are using the assumption that π is generically one to one to say

$$\#\left(\Lambda \cap \pi_p X\right) = \#\left(\overline{p\Lambda} \cap X\right)$$

as the general plane Λ will then miss the closed subset where the map is not one to one.

In general, we will have

$$\deg \pi_p \# (\Lambda \cap \pi_p X) = \# \left(\overline{p\Lambda} \cap X\right)$$

So, in general, $\deg \pi_p(X) = \frac{\deg X}{\deg \pi_p}$.

37. Proof of Bezout's Theorem

We'll start with a discussion of joins, which is an extension of the notion of cones.

Definition 37.1. Let X and Y be two irreducible projective varieties in $\mathbb{P}^n$. Say

$$\dim X = k$$
$$\dim Y = l.$$

Suppose further that $X \cap Y = \emptyset$. Then, the **join** of X and Y is

$$J(X, Y) = \cup_{x \in X, y \in Y} \overline{xy}.$$

This is a subvariety of $\mathbb{P}^n$ since we have a regular map

$$\phi : X \times Y \to \mathbb{G}(1, n)$$
$$(x, y) \mapsto \overline{xy}$$

where this map is given by the minors of the matrix of coordinates of x and y (the locus where the matrix has rank 1). Let $\mathcal{J} \subset \mathbb{G}(1, n)$ be the image. Then,

$$J = \cup_{\ell \in J} \ell \subset \mathbb{P}^n.$$

Lemma 37.2. *Suppose that $X, Y \subset \mathbb{P}^n$ are irreducible projective varieties of dimensions k and l with $X \cap Y = \emptyset$, and suppose further that a general point of J lies on only finitely many lines $\overline{xy}$ for $x \in X, y \in Y$. Then,* $\dim J = k + l + 1$.

Proof. We can use this to understand $\dim J$. The assumption that X and Y are disjoint implies that the regular map ϕ constructed in Definition 37.1 is finite. Therefore, in this case, $\dim \mathcal{J} = k + l$. Further, if a general point of J lies on only finitely many lines $\overline{xy}$ for $x \in X, y \in Y$, then $\dim J = k + l + 1$ ☐

Question 37.3. We have found the dimension of J. What is the degree of J?

In order to answer this question, we will first need a sizable amount of setup.

Suppose $\mathbb{P}^n = \mathbb{P}V$, with $\dim V = n+1$. Consider the two inclusions

$$\mathbb{P}^n \hookrightarrow \mathbb{P}(V \oplus V) \cong \mathbb{P}^{2n+1}.$$

We then have two copies of $\mathbb{P}^n$ in $\mathbb{P}^{2n+1}$ spanning $\mathbb{P}^{2n+1}$. Call these two copies of $\mathbb{P}^n$ Λ_1 and Λ_2. For concreteness, we may take

$$\Lambda_1 = V(z_{n+1}, \ldots, z_{2n+1})$$
$$\Lambda_2 = V(z_0, \ldots, z_n).$$

Let

$$\widetilde{X} := \operatorname{im} X \hookrightarrow \Lambda_1 \subset \mathbb{P}^{2n+1}$$
$$\widetilde{Y} := \operatorname{im} Y \hookrightarrow \Lambda_1 \subset \mathbb{P}^{2n+1}.$$

Inside $\mathbb{P}^{2n+1}$, there is a third copy of $\mathbb{P}^n$, call it Γ, where $\Gamma = \mathbb{P}(\Delta)$, where Δ is the diagonal. In coordinates, $\Gamma = V(z_0 - z_{n+1}, \ldots, z_n - z_{2n+1})$. Note that Γ is disjoint from X and from Y. Further,

$$\pi_\Gamma : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^n$$

is a rational map defined outside of Γ. In particular, it is defined on Λ_1 and Λ_2, so that

$$\pi_\Gamma|_{\Lambda_1} : \Lambda_1 \cong \mathbb{P}^n$$
$$\pi_\Gamma|_{\Lambda_2} : \Lambda_2 \cong \mathbb{P}^n$$
$$\pi_\Gamma|_{\widetilde{X}} : \widetilde{X} \cong X$$
$$\pi_\Gamma|_{\widetilde{Y}} : \widetilde{Y} \cong Y$$

Set

$$\widetilde{J} := J(\widetilde{X}, \widetilde{Y}) \subset \mathbb{P}^{2n+1}.$$

Recall we have

$$\begin{aligned} S := S(\mathbb{P}^{2n+1}) &:= \Bbbk\,[z_0, \ldots, z_{2n+1}] \\ &= \Bbbk[z_0, \ldots, z_n] \otimes_{\Bbbk} \Bbbk[z_{n+1}, \ldots, z_{2n+1}], \end{aligned}$$

and so this ring can be bigraded by $S_{i,j}$ defined to be the set of bihomogeneous polynomials $f(z_0, \ldots, z_{2n+1})$ which are bihomogeneous of bidegree (i,j) in the two sets of variables $(z_0, \ldots, z_n)$, $(z_{n+1}, \ldots, z_{2n+1})$. If $f \in \Bbbk[z_0, \ldots, z_{2n+1}$ has homogeneous degree m, we can write

$$f = g_0 + \cdots + g_m,$$

where g_i has bihomogeneous degree $(i, m - i)$.

Lemma 37.4. *Using the notation above we have* $f \in I(\widetilde{J}) \iff g_i \in I(\widetilde{J}$ *for all* i.

Proof. In general, a polynomial vanishes if and only if each graded piece vanishes. $\square$

Lemma 37.5. *We have*

Proof. By Lemma 37.4, we have

$$A_m(\widetilde{J}) = \oplus_{i+j=m} A_i(\widetilde{X}) \otimes A_j(\widetilde{Y}).$$

$\square$

We will now use the above lemmas to express the Hilbert function of the join J in terms of the Hilbert function of X and Y.

Lemma 37.6. *We have*

$$\sum_{i+j=m} \binom{a+i}{a}\binom{b+j}{b} = \binom{a+b+m+1}{m}.$$

Proof. We'll see this on Friday. $\square$

We can then use this to calculate $h_X(i) \cdot h_Y(j)$, which we will do on Friday.

38. 4/22/16

38.1. **Overview.** There are three classes left! Today, we'll wrap up Bezout's theorem via the calculation of joins. We'll also see a direct computation of the first nontrivial case of intersecting two curves in a plane. We'll also discuss the strong Bezout's theorem

38.2. **Binomial identities.**

Definition 38.1. Define

$$\binom{a+i}{i} := \frac{(a+i)(a+i-1)\cdots(a+1)}{i!}$$

Exercise 38.2. Show that, using the above definition of binomial coefficient, show the following three quantities are equal.

(1) $\binom{a+i}{i}$
(2) $(-1)^i\binom{-a-1}{i}$
(3) The coefficient of t^i in $(1+t)^{a+i}$.

Lemma 38.3. *We have*

$$\sum_{i+j=m} \binom{a+i}{a}\binom{b+j}{b} = \binom{a+b+m+1}{m}.$$

Proof. Indeed, expanding, we see

$$\begin{aligned}\sum_{i+j=m} \binom{a+i}{i}\binom{b+j}{j} &= \sum_{i+j=m} \binom{-a-1}{i}\binom{-b-1}{j}(-1)^{i+j} \\ &= (-1)^m \binom{-a-b-2}{m} \\ &= \binom{a+b+1+m}{m}.\end{aligned}$$

□

38.3. **Review of the setup from the previous class.** Recall our setup from last class. Let $X, Y \subset \mathbb{P}^n$ be subvarieties of $\mathbb{P}^n$ of dimensions k and l respectively, and degrees d and e respectively.

The trick for proving Bezout's theorem is by embedding $\mathbb{P}^n$ in $\mathbb{P}^{2n+1}$ in two different ways. That is, if $\mathbb{P}^n \cong \mathbb{P}V$, we embed $V \to V \oplus V$ as the two direct summands, corresponding to two maps $\mathbb{P}^n \cong \mathbb{P}V \to \mathbb{P}(V \oplus V) \cong \mathbb{P}^{2n+1}$. We also have an n-plane Γ in $\mathbb{P}^{2n+1}$ corresponding to the diagonal map. Concretely in coordinates, we have two maps whose images are

$$\phi_1 : \mathbb{P}^n \to \Lambda_1 = V(z_{n+1}, \ldots, z_{2n+1}) \subset \mathbb{P}^{2n+1}$$
$$\phi_2 : \mathbb{P}^n \to \Lambda_2 = V(z_0, \ldots, z_n) \subset \mathbb{P}^{2n+1}.\phi_3 : \mathbb{P}^n \to \Gamma \quad = V(z_0 - z_{n+1}, \ldots, z_n - z_{2n+1}).$$

Define

$$\pi_\Gamma : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^n$$

to be the projection away from Γ. Take

$$\begin{aligned}\widetilde{X} &:= \phi_1(X) \\ \widetilde{Y} &:= \phi_2(Y)\widetilde{J} \qquad := J(\widetilde{X}, \widetilde{Y}).\end{aligned}$$

where $J(A, B)$ is the join of A and B, defined last time as the variety connecting all pairs in $A \times B$ by lines. As we saw last time, we can bigrade the homogeneous coordinate ring

$$S(\mathbb{P}^{2n+1})_m = \oplus_{i+j=m} S(\Lambda_1)_i \otimes S(\Lambda_2)_j.$$

More concretely, we are just writing a polynomial in all the variables as a polynomial in the two sets of variables (those up to n and those bigger than n). That is, the map above is given by writing

$$f(z_0, \ldots, z_{2n+1}) = \sum_{i+j=m} G_i(z_0, \ldots, z_n) \cdot H_j(z_{n+1}, \ldots, z_{2n+1})$$

where G_i has homogeneous degree i and H_j has homogeneous degree j.

38.4. **Proving Bezout's theorem.**

Lemma 38.4. *Recall that X and Y are varieties in $\mathbb{P}^n$ of degrees d and e and dimensions k and l. Using $\widetilde{J}$ defined above, we have We have*

$$\deg \widetilde{J} = de$$

Proof. Observe that since

$$S(\mathbb{P}^{2n+1})_m = \oplus_{i+j=m} S(\Lambda_1)_i \otimes S(\Lambda_2)_j.$$

we also have

$$S(\widetilde{J}) = \oplus_{i+j=m} \left(S(X)_i \otimes S(Y)_j \right).$$

This directly translates to

$$h_{\widetilde{J}}(m) = \sum_{i+j=m} h_X(i) h_Y(j).$$

Now, expanding this, further, we have

$$\begin{aligned} h_{\widetilde{J}}(m) &= \sum_{i+j=m} h_X(i) h_Y(j) \\ &= \sum_{i+j=m} \left(d \cdot \frac{i^k}{k!} + O(i^{k-1}) \right) \left(e \frac{i^l}{l!} + O(i^{l-1} \right) \\ &= \sum_{i+j=m} \left(d \binom{i+k}{i} + \cdots \right) \left(e \binom{j+l}{j} + \cdots \right) \\ &= de \left(\binom{m+k+l+1}{m} \right) + O(m^{k+l}) \\ &= de \frac{m^{k+l+1}}{(k+l+1)!} + \cdots, \end{aligned}$$

where for going from the third line to the fourth line, we are using Lemma 38.3. In particular,

$$\deg \widetilde{J} = de.$$

□

We observe a corollary that will not be used in the case of Bezout's theorem.

Corollary 38.5. *Suppose* $\pi_\Gamma|_{\widetilde{J}} : \widetilde{J} \to J$ *is generically one to one and* X *and* Y *are disjoint. Then,* $\deg J = de$*.*

Proof. If X and Y are disjoint, then Γ will not meet $\widetilde{J}$. We are also using that a general point in $\overline{xy}$ for $x \in X$ and $y \in Y$ lies only on one such line. This is equivalent to the map being generically one to one. Then, since $\deg \widetilde{J} = de$, from Lemma 38.4 we also have $\deg J = de$. □

Start by recalling Bezout's theorem:

Theorem 38.6 (Bezout's Theorem)**.** *Say* $X, Y \subset \mathbb{P}^n$ *are irreducible projective varieties of dimensions* k, l *with* $k + l \geq n$*. If* $X \cap Y$ *is generically transverse then*

$$\deg X \cap Y = \deg X \cdot \deg Y.$$

Proof of Bezout's theorem. Observe that $\widetilde{J} \cap \Gamma \cong X \cap Y$.

Exercise 38.7. Show that X is generically transverse to Y if and only if $\widetilde{J}$ is generically transverse to Γ. *Possible hint:* The tangent space to the diagonal is a linear subspace, so we want a description of the tangent space to $\widetilde{J}$. Since X and Y have dimensions k and l, the tangent space to X at a smooth point is a $\mathbb{P}^k \subset \mathbb{P}^n$ and the tangent space to Y is a $\mathbb{P}^l \subset \mathbb{P}^n$. The tangent space of the join will be the span of the tangent space to X, $\mathbb{P}^k \subset \Lambda_1$ and the tangent space to Y, $\mathbb{P}^l \subset \Lambda_2$. Use this to deduce the claimed transversality.

Hence, if X is generically transverse to Y, then

$$\begin{aligned} \deg X \cap Y &= \deg \widetilde{J} \cap \Gamma \\ &= \deg \widetilde{J} \\ &= \deg X \cdot \deg Y, \end{aligned}$$

where the last equality uses Lemma 38.4. □

38.5. **Strong Bezout.**

Remark 38.8. We're not going to prove strong Bezout's theorem, but it is a fascinating story in the development of algebraic geometry.

Let's start by examining the simplest case of Bezout's theorem.

Corollary 38.9. *Say* $C, D \subset \mathbb{P}^2$*, with* $C = V(f), D = V(g)$ *with* $\deg f = d, \deg g = e$*, then Bezout's theorem says that if* C *intersects* D *transversely, we have* $\#(C \cap D) = de$*. This is the same as the number of common solutions to* $f = g = 0$*.*

Proof. Just apply Bezout's theorem in the case that the two varieties X and Y are curves C and D in $\mathbb{P}^2$. □

Remark 38.10. This is the two variable version of the fundamental theorem of algebra, which says that one polynomial in one variable has as many roots as its degree. Note that Bezout's theorem is saying two polynomials in two variables has as many solutions as the product of their degree.

Remark 38.11. In the fundamental theorem of algebra, we say that a degree n polynomial over the complex numbers has precisely n roots, counting multiplicity. We can now ask how to count multiplicity of intersections of two curves in $\mathbb{P}^2$.

We'll now see an alternate proof of Bezout's theorem for the case of two curves in projective space.

Proposition 38.12. *Say* $C, D \subset \mathbb{P}^2$*, with* $C = V(f), D = V(g)$ *with* $\deg f = d, \deg g = e$*. If* C *and* D *intersect generically transversely, then* $\#(C \cap D) = de$*.*

Proof. Write

$$f(x, y, z) = \sum_i a_i(x, y) z^i$$
$$g(x, y, z) = \sum_j b_j(x, y) z^j.$$

Observe that F and G have a common zero if and only if the resultant $R(x, y) = 0$, where $R(x, y)$ is the determinant

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_d & 0 & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{d-1} & a_d & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 & \cdots & a_d \\ b_0 & b_1 & \cdots & b_e & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & b_{e-1} & b_e & 0 & 0 & \cdots & 0 \\ 0 & b_0 & \cdots & b_{e-1} & b_e & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_e \end{pmatrix}$$

This is a homogeneous polynomial in x and y of degree $d \cdot e$.

Exercise 38.13 (Difficult exercise)**.** Verify that C and D are transverse at (x, y) if and only if $R(x, y)$ has a simple zero at (x, y).

□

Remark 38.14. However, the resultant gives us an idea for how to generalize Bezout's theorem for non-transverse intersections, and we now define intersection multiplicity.

In fact, we can define the intersection multiplicity of C and D at $p \in C \cap d$ to be a positive integer $m_p(C \cdot D)$. This has the property that $m_p(C \cdot D) = 1$ if and only if C is generically transverse at p. In the special case of plane curves, this is the degree of zero of the resultant. We have, a stronger form of Bezout's theorem in this case, which says

$$m_p(C \cdot D) = de,$$

counting multiplicity, whenever $\#(C \cap D) < \infty$.

We now state a strong form of Bezout's theorem, which is generalization of the case of plane curves.

Theorem 38.15 (Strong Bezout)**.** *Assume that X and Y are irreducible varieties and that $\dim X \cap Y = \dim X + \dim Y - n$. We can assign to any component Z of a proper intersection $X \cap Y \subset \mathbb{P}^n$ a multiplicity*

$$m_Z(X \cdot Y)$$

so that

$$\sum m_Z(X \cdot Y) \cdot \deg Z = \deg X \cdot \deg Y.$$

Proof. Omitted □

Remark 38.16. We importantly haven't given the formula. This wasn't found until the 1950's by Serre, who characterized the multiplicity in terms of Tor functors.

39. 4/25/16

39.1. **Overview.** There are only two classes left. We'll talk about some topics which won't be tested on the homework. Today, we'll talk about real plane curves.

39.2. **The question of connected components over the reals.** The original problem motivating algebraic geometry was the following.

Question 39.1. Describe the zero locus of a polynomial $f(x, y) \in \mathbb{R}[x, y]$ of degree d as a subset of $\mathbb{R}^2$.

Example 39.2. In the case that $d = 2$ (f is of degree 2), the possible zero loci are hyperbolas, parabolas, ellipses, and the empty set.

One natural follow up question is the following.

Question 39.3. How many connected components can the zero locus of $f(x, y) \in \mathbb{R}[x, y]$ have?

Remark 39.4. In classical 19th century language, the connected components were called "ovals."

Example 39.5. When $d = 3$, the possible cubics may look like $y^2 = x^3 - x$ (or, more generally, an elliptic curve with positive discriminant), which have two connected components, $y^2 = x^3 + x$ (or, more generally, an elliptic curves with negative discriminant), which have one component. It's also possible to have a cubic with three components by changing variables for the second cubic. One can also have cubics which are singular, a subset of which are the unions of a line with a conic.

Today, we'll describe how to find the number of components. After people started in algebraic geometry, people came along and said we shouldn't be working over $\mathbb{R}$, but should instead be working over $\mathbb{C}$, and be using projective space. That said, there was an implicit promise that after they worked over projective space, they would later come back and answer the question about real curves.

This promise has not always been kept, but today we will keep it!

39.3. **The genus is constant in families.** We'll assume that $f(x, y) \in \mathbb{R}[x, y]$ is smooth. We're trying to find the number of connected components.

If we start by passing from $\mathbb{R}^2$ to $\mathbb{P}^2_{\mathbb{C}}$. The corresponding zero locus $f(x, y, z) = 0$ is a compact, connected, oriented (since it is complex) 2-manifold.

We have a topological classification of these manifolds as a g-holed torus. The question is:

Question 39.6. What possible genera arise as the genus of a smooth plane curve?

The answer turns out to be any genus of the form $\binom{d-1}{2}$, and we'll see this later today. We start by showing the important fact that the genus is the same for all degree d curve.

Lemma 39.7. *Let f, g be two smooth plane curves in $\mathbb{P}^2_{\mathbb{C}}$ of degree d. Then, f and g have the same (topological) genus.*

Proof. Recall that we have a universal family of degree d curves

$$\mathcal{C} := \{(C, p) : p \in C\} \subset \mathbb{P}^N \times \mathbb{P}^2,$$

where $N = \binom{d+2}{2} - 1$. More concretely, $\mathcal{C} = V(\sum_{ijk} a_{ijk} x^i y^j z^k)$. We have projections

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathbb{P}^N & & & & \mathbb{P}^2. \end{array} \tag{39.1}$$

Inside $\mathbb{P}^N$ we have an open subset $U \subset \mathbb{P}^N$, so that the fibers of π_1 are smooth curves. (Formally, U can be defined as the complement of the image of the singular locus of the map π_1, though we haven't defined singular locus.) We define $\mathcal{C}_U$ as the fiber product (i.e., the preimage of U)

$$\begin{array}{ccc} \mathcal{C}_U & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathbb{P}^N. \end{array} \tag{39.2}$$

So, we have an projections

$$\begin{array}{ccccc} & & \mathcal{C}_U & & \\ & \swarrow^{\eta_1} & & \searrow^{\eta_2} & \\ U & & & & \mathbb{P}^2. \end{array} \tag{39.3}$$

where $\eta_i := \pi_i|_{\mathcal{C}_U}$.

Suppose c_0 and c_1 are two points in U. Since $\mathcal{C}_U, U$ are both smooth, and the map $\mathcal{C}_U \to U$ is a submersion, we obtain that all fibers are homeomorphic. In fact, $\mathcal{C}_U \to U$ is a fiber bundle. So, all fibers have the same genus. $\square$

Remark 39.8. The above argument may fail because the smooth locus $U \subset \mathbb{P}^N$ may not be connected. If one removes a hypersurface from real projective space the resulting U may have a large number of connected components (instead of just 1 in the complex case). However, the number of connected components of U is still unknown.

39.4. **Finding the genus.** We'll next try to find the genus of a degree d curve.

Example 39.9. One of the big motivations for studying these questions was the calculus for integrals of algebraic functions. In the case of degree 2, the curve is a sphere, so the genus is 0. When $d = 3$, the genus is 1.

Proposition 39.10. *Let C be a degree d smooth curve in $\mathbb{P}^2_{\mathbb{C}}$. Then, C has genus $\binom{d-1}{2}$.*

Proof. Start with a union of lines in $\mathbb{P}^2$, by which we mean, topologically, a union of spheres.

We set $g = \prod_i \ell_i(x, y)$. Now, set $f = g + \varepsilon$, and we will try to find the genus of $V(f)$. We'll have a collection of spheres $V(g)$, which is a crazy connection of spheres meeting at some points, and then we'll smooth it.

Let's start in the simplest case of $xy = 0$. This is two disks meeting at a point.

In the complex setting $xy = \varepsilon$, one will see a cylinder, replacing the two disks with a cylinder.

In general, we will replace all the meeting points of disks by such a cylinder.

To find the genus, we use the topological invariant, which is the Euler characteristic.

Let $V(g)$ be the curve X_0 and let $V(f)$ be called X_1. It suffices to calculate the Euler characteristic of X_0 is, as this will be the same as the Euler characteristic of X_1, and hence the Euler characteristic of any degree d curve by Lemma 39.7.

Now, a sphere has Euler characteristic 2, so a disjoint union of d spheres has Euler characteristic $2d$. When we identify two lines at a point, the Euler characteristic drops by 1, since the number of points drop by 1.

So, the Euler characteristic of X_0 is equal to $2d$, minus the number of intersection points, which is

$$\chi(X_0) = 2d - \binom{d}{2}.$$

Next, note that when we replace two disks meeting at a point by a cylinder, we start with an object of Euler characteristic 1, since this union is contractible. So, we see the Euler characteristic of a cylinder is 0. Hence, using Mayer-Vietoris, we see that these replacements decrease the Euler characteristic by 1. In total, we have that the Euler characteristic of X_1 is

$$\chi(X_1) = 2d - \binom{d}{2} - \binom{d}{2}. = -d(d-3).$$

Note that

$$2 - 2g = \chi(X_1) = -d(d-3).$$

Therefore,

$$g = \binom{d-1}{2}.$$

□

39.5. **Finding the number of connected components of $\mathbb{R}$.** By Proposition 39.10,

we know a smooth complex curve of degree d has genus $\binom{d-1}{2}$. We now want to find the number of connected components over $\mathbb{R}$. Let $f(x, y, z) \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of degree d.

Let

$$C_{\mathbb{R}} := V(f) \subset \mathbb{P}^2_{\mathbb{R}}$$
$$C_{\mathbb{C}} := V(f_{\mathbb{C}}) \subset \mathbb{P}^2_{\mathbb{C}}.$$

We want to know the number of connected components of $C_{\mathbb{R}}$. To pass from the case of projective space to that of affine space, we're just asking for the number of ovals in the complement of a line in projective space, so it suffices to answer the question for the case of projective space, which we now do.

Theorem 39.11 (Hannack's theorem)**.** *A smooth projective curve over the reals has at most*

$$\binom{d-1}{2} + 1$$

components.

Proof. We know that $C_{\mathbb{C}}$ is a compact oriented two manifold of genus g. To find the number of connected components of $C_{\mathbb{R}}$, we note that we have a continuous involution

$$\begin{aligned} \tau : C_{\mathbb{C}} &\to C_{\mathbb{C}} \\ [x, y, z] &\mapsto [\overline{x}, \overline{y}, \overline{z}], \end{aligned}$$

which is a continuous orientation reversing map.

Exercise 39.12. Show that the fixed point set of $C_{\mathbb{C}}$ under τ is precisely $C_{\mathbb{R}}$.

Question 39.13. What is the quotient $C_{\mathbb{C}}/\tau$?

The resulting quotient $C_{\mathbb{C}}/\tau$ is a 2-manifold with boundary. The boundary is precisely $C_{\mathbb{R}}$. So, if $C_{\mathbb{R}}$ has δ connected components, we can complete this to a compact two manifold $C_{\mathbb{C}}/\tau$ by adding in δ two-disks. Let the resulting compact manifold be $\overline{C}$. We know

$$\chi(C_{\mathbb{C}}) = -d(d-3).$$

After removing $C_{\mathbb{R}}$, since a circle has Euler characteristic 0, we have

$$\chi(C_{\mathbb{C}} \setminus C_{\mathbb{R}}) = \chi(C_{\mathbb{C}}) = -d(d-3).$$

Then, the action of τ on $C_{\mathbb{C}} \setminus C_{\mathbb{R}}$ is fixed point free, so we have

$$\chi(C_{\mathbb{C}} \setminus C_{\mathbb{R}})/\tau = \frac{d(d-3)}{2}.$$

Hence, letting $\overline{C}$ be the addition of δ disks, as described above.

$$\chi(\overline{C}) = \frac{-d(d-3)}{2} + \delta.$$

This is at most 2, since it is a compact connected two manifold with some nonnegative genus. Finally, we see

$$\delta \leq \binom{d-1}{2} + 1$$

□

Remark 39.14. In fact, the bound

$$\delta \leq \binom{d-1}{2} + 1$$

is sharp, as can be seen by explicitly exhibiting such a curve. And further, every possible number of connected components from 1 up to $\binom{d-1}{2} + 1$ appears.

d	number of connected components
1	1
2	1
3	2
4	4
5	7
6	11

TABLE 4. The number of connected components of curves of low degree

This story is an illustration of how mathematics works. Hundreds of years ago, people asked how to find the number of connected components. People went to complex projective space to answer the question. This then gave us the information to deduce the answer to our original question.

40. 4/27/16

40.1. **Finishing up with the number of connected components of real plane curves.** Let $C \subset \mathbb{P}^2_{\mathbb{R}}$ be a smooth real plane curve of degree d. Last time, we found the number of connected components of C is at most

$$\binom{d-2}{2} + 1.$$

Making a table of the number of connected components by degree, we have

Example 40.1. Here's an example of how to construct a degree 4 curve with four connected components. Take a product of two conics $q_1 q_2$, and then disturb it by ε, so that we have an equation of the form $q_1 q_2 + \varepsilon$. This is a degree 4 curve, and you can see it as near the boundary of four holes in the intersection of two ellipses.

Remark 40.2. Coolidge's book on "higher plane curves" has a nice fold out constructing examples of curves like this.

40.2. **Parameter spaces.** Today, we'll look at parameter spaces for varieties in $\mathbb{P}^n$, vaguely following Chapter 21.

Goal 40.3. Given a class $\mathcal{C} = \{X \subset \mathbb{P}^n\}$ of projective varieties we want a bijection between the points of a variety $\mathcal{H}$, which is a parameter space. That is, we want a closed subvariety

(40.1)
$$\begin{array}{ccc} \Phi := \{([X], p) : p \in X\} & \xrightarrow{\iota} & \mathcal{H} \times \mathbb{P}^n \\ & \searrow \quad \swarrow & \\ & \mathcal{H} & \end{array}$$

with ι realizing Φ as a closed subvariety of $\mathcal{H} \times \mathbb{P}^n$. (In a more advanced, precise setting, we would require that the left map is flat, though we have not defined flatness in this course.)

Example 40.4. Take

$$\mathcal{C} := \{ \text{ hypersurfaces of degree } d \text{ in } \mathbb{P}^n \}.$$

We have an inclusion

$$\mathcal{C} \cong U \subset \mathbb{P}^N = \{ \text{ homogeneous polynomials of degree } d \text{ on } \mathbb{P}^n \} / \{ \text{ scalars } \}.$$

This open subset U is precisely the subset corresponding to square free polynomials. Then, we have our parameter space

$$\Phi = V(\sum_I a_I Z^I) \subset U \times \mathbb{P}^n.$$

Question 40.5. How do we create parameter spaces for other classes of varieties.

Example 40.6 (Extended example: a parameter space for twisted cubics)**.** Let's try to construct a parameter space for twisted cubics $C \subset \mathbb{P}^3$.

40.2.1. *Unsuccessful attempt 1.* Recall that $C = \operatorname{im}(\mathbb{P}^1 \hookrightarrow \mathbb{P}^3)$ where the map $\mathbb{P}^1 \to \mathbb{P}^3$ is given by a four tuple of cubic polynomials which form a basis. However, this doesn't work, because many different 4-tuples of polynomials will give the same curve. For instance, if we compose with an automorphism of $\mathbb{P}^1$, we get the same curve.

40.2.2. *Unsuccessful attempt 2.* Another way to specify C is as $C = V(Q_1, Q_2, Q_3)$, and then take a space parameterized by the coefficients of the three Q_i. However, this again depends on the choice of the basis, as any two triples of quadratics generating the same three dimensional subspace of quadric polynomials will define the same curve.

40.2.3. *Successful attempt 3.* We already know how to get parameter spaces for hypersurfaces. Let's try to reduce our problem to this. If we have a twisted cubic curve in $\mathbb{P}^3$, we can associate a certain incidence variety in a grassmannian. That is, we can associate to C the locus of lines in $\mathbb{P}^3$ that meet C. That is, define

$$\Sigma_C := \{[\ell] \in \mathbb{G}(1,3) : \ell \cap C \neq \emptyset\}.$$

This is a hypersurface in the grassmannian. To see this, introduce

$$\Phi := \{(\ell, p) : p \in \ell \cap C\} \subset \mathbb{G}(1,3) \times C.$$

We have projections

$$\begin{array}{ccc} & \Phi & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \Sigma_C \subset \mathbb{G}(1,3) & & C. \end{array} \tag{40.2}$$

The fibers of π_2 are isomorphic to $\mathbb{P}^2$, and so we get Φ is a threefold, and one can also check that Σ_C is a threefold, as the fibers of π_1 are generically finite, since C does not contain any lines. Similarly, we can parameterize lines as a subspace of a product of projective spaces. To do this, consider

$$\Psi := \{(H_1, H_2, p) : p \in H_1 \cap H_2 \cap C\} \subset (\mathbb{P}^3)^\vee \times (\mathbb{P}^3)^\vee \times C.$$

We have projections

$$\begin{array}{ccc} & \Psi & \\ {}^{\eta_1}\swarrow & & \searrow^{\eta_2} \\ \left(\mathbb{P}^3\right)^\vee \times \left(\mathbb{P}^3\right)^\vee & & C \end{array} \tag{40.3}$$

Defining the image of η_1 to be Γ_C, we can conclude that Γ_C is a hypersurface in $\left(\mathbb{P}^3\right)^\vee \times \left(\mathbb{P}^3\right)^\vee$. If we fix H_1 and let H_2 vary along a line in $\left(\mathbb{P}^3\right)^\vee$, meaning H^2 varies over planes containing a line not contained in H_1. Since H_1 is a general plane, it will meet C in three points, by Bezout's theorem. As we vary H_2, there will be three values where the intersection is nonempty. So, Γ_C is the zero locus of a bihomogeneous polynomial of bidegree $(3,3)$ on $\left(\mathbb{P}^3\right)^\vee \times \left(\mathbb{P}^3\right)^\vee$. If we were looking at curves of degree d rather than the twisted cubic, we would get a hypersurface of bidegree (d, d).

We can associated to C the curve $[f]$ with $\Gamma_C = V(f)$. Here,

$$f \in \mathbb{P}\left(\text{bihomogeneous polynomials of bidegree } (3,3) \text{ on } \left(\mathbb{P}^3\right)^\vee \times \left(\mathbb{P}^3\right)^\vee\right).$$

So, we have constructed twisted cubics as a subspace of $\mathbb{P}^{399}$, where this is the projectivization of a 400 dimensional vector space, where $400 = 20 \cdot 20$, with 20 the dimension of cubic polynomials on $\mathbb{P}^3$.

40.3. **Successful Attempt 4.** We now give yet another possible construction for the parameter space of twisted cubics, which works more generally and is due to Grothendieck. This is the essential idea in the construction of the Hilbert scheme.

Recall our second unsuccessful attempt: We tried to choose three quadrics. But, instead, we could try to chose a three dimensional subspace of quadrics. This is a three dimensional subspace of the 10 dimensional vector space of quadrics on $\mathbb{P}^3$, which is a point of the grassmannian $G(3,10)$.

However, not all three dimensional subspaces of $G(3,10)$ will cut out a twisted cubic, since an intersection of three general quadrics will only be 8 isolated points, using Bezout's theorem. So, we want to realize the locus of three dimensional subspace of quadrics on $\mathbb{P}^3$, which do indeed cut out a twisted cubic.

We get an injection

$$\{\text{ twisted cubics }\} \hookrightarrow G(3,10).$$

To see this is locally closed, define

$$\Phi := \left\{(\Lambda, p) \in G(3,10) \times \mathbb{P}^3 : Q(p) = 0 \text{ for all } Q \in \Lambda\right\} \subset G(3,10) \times \mathbb{P}^3.$$

We then have projections

(40.4)
$$\begin{array}{ccc} & \Phi & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ G(3,10) & & \mathbb{P}^3 \end{array}$$

This gives a map $\Phi \to G(3,10)$ and an open subset of an irreducible component of the locus where π_1 has 1 dimensional fibers gives a parameter space for twisted cubics as a locally closed subset of $G(3,10)$.

Remark 40.7. We have proven the existence of a parameter space satisfying the requirements we wanted, but to write it down explicitly takes a lot more work.

The first successful construction was Chow's, the second successful construction was Grothendieck's, which has essentially superseded Chow's construction, as it works scheme theoretically.

Remark 40.8. If we try to generalize Grothendieck's approach a bit, beyond twisted cubics, we run into some technical issues. If we start with a general family of curves, we want to look at these curves as corresponding to a certain subspace of a grassmannian. A natural set to consider is

$$\mathcal{C} := \{X \subset \mathbb{P}^n : X \text{ has Hilbert polynomial } p\}.$$

We want to associate to $X \in \mathcal{C}$ the subset of the grassmannian

$$(I(X)_m \subset S(\mathbb{P}^n)_m) \in G\left(\binom{m+n}{n} - h_X(m), \binom{m+n}{n}\right).$$

In order for this to apply to all subvarieties with a given Hilbert polynomial, we need to know two things:

(1) Given a Hilbert polynomial p, there exists m_0 so that for all $m > m_0$ and for all $X \subset \mathbb{P}^n$ with Hilbert polynomial p, we have
$$h_X(m) = p_X(m).$$
(2) Given a Hilbert polynomial p, there exists m_0 so that for all $m > m_0$ and for all $X \subset \mathbb{P}^n$ with Hilbert polynomial p, we want $X = V(I(X)_m)$.

Recall that for both of these points, we know they both hold for sufficiently large m with X fixed, but we want this bound to also be independent of X. This takes a significant amount of work, and these properties were proven by Matsusaka in the 1950s. Grothendieck's' idea was to use this result to give a construction of the Hilbert scheme.