Math 6421

Online or in person fine.

Some HW per week (drop 3) Gradescope.

Online office hours Fri 1:30-2:30 Teams
& by appt.

Overview of Course

What is alg. geom?

study of solns of polynomials

(using ring theory, etc.)

"linear alg, without the linear"

much harder.

Setup

k = field (usually can think $k = \mathbb{C}$)

$k[x_1, \dots, x_n]$ = ring of polynomials in $x_1, \dots, x_n$ with coeffs in k.

$\mathbb{A}^n = \mathbb{A}^n_k$ = affine n-space over k

$= \{(a_1, \dots, a_n) \in a_i \in k\}$

In bijection with k^n. In $\mathbb{A}^n$ no vect. sp structure, so 0 not special etc.

For $f_1, \dots, f_r \in k[x_1, \dots, x_n]$:

$$Z(f_1, \dots, f_r) = \{(a_1, \dots, a_n) \in \mathbb{A}^n : f_i(a_1, \dots, a_n) = 0 \ \forall i\}$$

zero set or vanishing set

Some texts use V instead of Z

These are affine algebraic varieties.

Special cases

① $n = r = 1$

d = degree of poly.

Solving polynomials in 1 var.

k alg closed : exactly d solns (with mult)

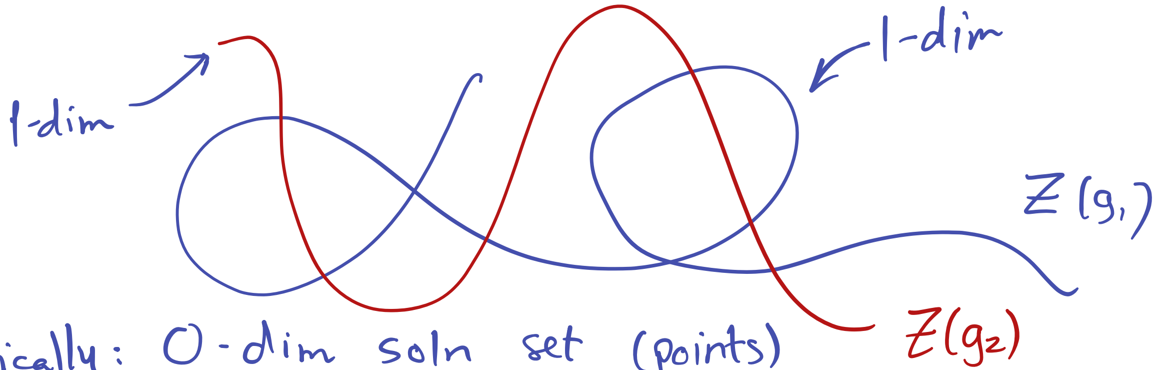
② Linear Algebra

Again, much harder when deg > 1 or # eqns > 1.

Example

$$g_1(x,y) = 3x^3 - 17xy^2 + 2xy + 4y^2 - 6$$
$$g_2(x,y) = x^5 - x^3y^2 + 3xy^2$$

We'll learn: $Z(g_i)$ is a "curve" in $\mathbb{C}^2$



Generically: 0-dim soln set (points)

Weak Bezout's Thm

If $f_1, f_2 \in \mathbb{C}[x,y]$, no common factors

$\deg f_i = d_i$

Then $|Z(f_1, f_2)| \leq d_1 d_2$.

Chapter 1 The geometry/algebra dictionary

$\boxed{\text{Alg} \to \text{Geom}}$ Given $J \subseteq k[x_1, \dots, x_n]$ ideal

$$\rightsquigarrow Z(J) = \{ a \in \mathbb{A}^n : f(a) = 0 \ \forall f \in J \}$$

example $J = \langle f_1, \dots, f_r \rangle$ ideal generated by $f_1, \dots, f_r$

$$= \{ g_1 f_1 + \dots + g_r f_r : g_i \in k[x_1, \dots, x_n] \}$$

Then $Z(J) = Z(f_1, \dots, f_r)$ as above.

$\boxed{\text{Geom} \to \text{Alg}}$ Given $V \subseteq \mathbb{A}^n$

$$\rightsquigarrow I(V) = \{ f \in k[x_1, \dots, x_n] : f(a) = 0 \ \forall a \in V \}$$

this is an ideal.

We have:

$$\{\text{subsets of } \mathbb{A}^n\} \rightleftarrows \{\text{ideals in } k[x_1, \dots, x_n]\}$$

Neither is injective. Why?

$\leftarrow$ $Z(x) = Z(x^2)$ more interesting direction

$\rightarrow$ all open sets in $\mathbb{C}$. $\longmapsto$ 0 ideal.

To fix latter, replace LHS with affine alg vars

For former, the example is the <u>only</u> issue. (taking powers).

The fix: For an ideal $J \subseteq R$, have

$$\mathrm{rad}(J) = \{ r \in R : r^i \in J \text{ some } i \geq 1 \}$$

"radical"

Will use Hilbert's Nullstullensatz to show

$$\{\text{affine alg var's in } \mathbb{A}^n\} \longleftrightarrow \{\text{radical ideals in } k[x_1, \dots, x_n]\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

also:

$$\{\text{irreducible alg vars in } \mathbb{A}^n\} \longleftrightarrow \{\text{prime ideals} \dots\}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\mathbb{A}^n = \{\text{pts in } \mathbb{A}^n\} \longleftrightarrow \{\text{maximal ideals}\}$$

Chapter 2 Projective Varieties.

$k-0$

$$\mathbb{P}^n = \mathbb{P}^n_k = \left(k^{n+1} - 0\right) / k^*$$

where $v \sim w$ if $v = cw$

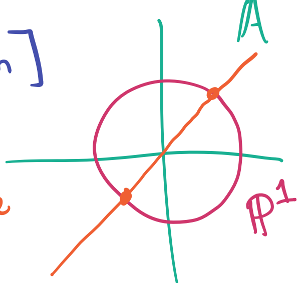
$c \in k^*$.

$=$ set of lines in k^{n+1}

Write equiv class of $(a_0, \dots, a_n)$

as $[a_0 : a_1 : \cdots : a_n]$

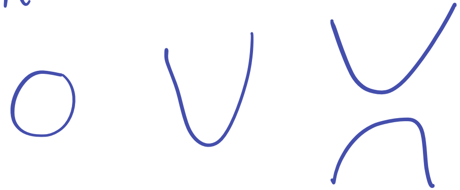
In $\mathbb{R}^2$:



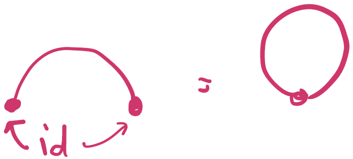
We will study zero sets in $\mathbb{P}^n$ ....

because: more symmetry.

In $\mathbb{A}^2$ have different conics

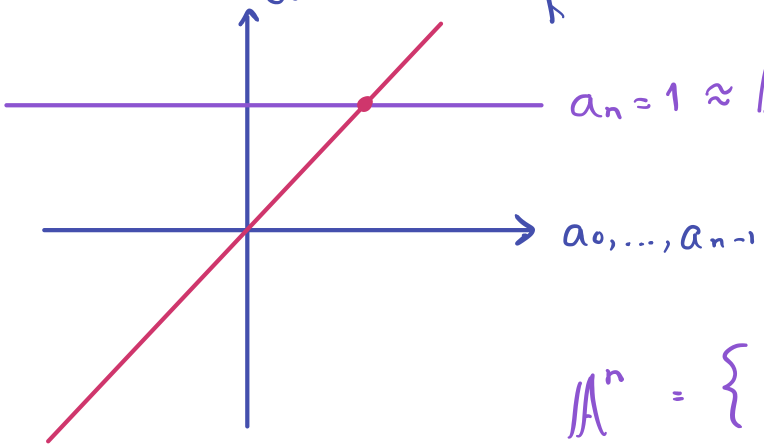


In $\mathbb{P}^2$ these are all same!



Can regard $\mathbb{P}^n$ as $\mathbb{A}^n \cup \mathbb{P}^{n-1}$.

k^{n+1}.



$$\mathbb{A}^n = \{ [a_0 : \cdots : a_n] : a_n \neq 0 \}$$

$$\mathbb{P}^{n-1} = \{ [a_0 : \cdots : a_{n-1} : a_n] : a_n = 0 \}$$

Projective varieties

$f(a_0, \ldots, a_n)$ is not well-defined on $\mathbb{P}^n$

e.g. $f(x,y) = x + y^2$

$f(-1,1) = 0$

$f(-2,2) = 2$

So can't say $f([-1:1]) = 0$.

But, if f is homogeneous (all terms have same degree d)

then $f(cv) = c^d f(v)$.

So $f(v) = 0 \iff f(cv) = 0$ ✓

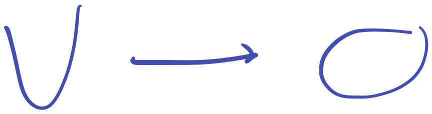
So: get zero sets in $\mathbb{P}^n$ for homog. poly's.

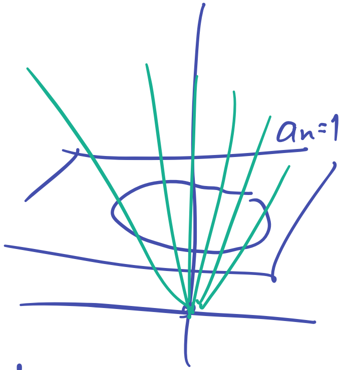
So for $f_1, \ldots, f_r$ homogeneous.

$$Z(f_1, \ldots, f_r) = \{a \in \mathbb{P}^n : f_i(a) = 0 \ \forall i\}$$

We'll see:

① Affine varieties have projective closures





② Cone on a proj. var. is an aff. variety.

So, the theories are closely related.

Next time: Better Bezout

$Z(f_1)$ curves in $\mathbb{P}^2$ of deg d_i

$Z(f_2)$ & f_1, f_2 no common factors.

Then $|Z(f_1) \cap Z(f_2)| = d_1 d_2$ (count with multiplicity).