Overview. From last time:

Chapter 1. The geometry/algebra dictionary

$f_1, \dots, f_r \in k[x_1, \dots, x_n]$

$Z(f_1, \dots, f_r) = \{a \in \mathbb{A}^n : f_i(a) = 0 \ \forall i\}$

"affine algebraic variety"

There is a bijection

$$\{\text{aav's in } \mathbb{A}^n\} \longleftrightarrow \left\{\begin{matrix}\text{rad ideals in} \\ k[x_1, \dots, x_n]\end{matrix}\right\}$$

$$V \longmapsto I(V)$$

$$Z(J) \longleftarrow J$$

Chapter 2. Projective varieties

$\mathbb{P}^n := (k^{n+1} - 0)/k^*$

$[(a_0, \dots, a_n)]$ written $[a_0 : \cdots : a_n]$

$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$

homogeneous poly's in $k[x_0, \dots, x_n]$

$\rightsquigarrow$ projective alg. var's

These are always compact and tend to have more symmetry/info (e.g. intersections at ∞).

Chapter 3 Classical constructions

① Segre embedding

$$\varphi_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$$

consequence: product of varieties is a variety

example: $g_1(x,y) = 3x^3 - 17xy^2$
$g_2(z,w) = z^5 - w^2 z^3$

Does $Z(g_1 g_2)$ work?

No. get $Z(g_1) \times \mathbb{P}^1$
$\mathbb{P}^1 \times Z(g_2)$
and more...

② Veronese embedding

$$v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{d+n}{n}-1}$$

reduces the degree

For example: "Fermat cubic curve"

$$Z(x_0^3 + x_1^3 + x_2^3) \subseteq \mathbb{P}^2$$

maps intersection of 9 quadrics in $\mathbb{P}^9$.

↳ zero sets of quadratic.

param space of all polys of $d=2$
$\{(a,b,c)\} = \mathbb{A}^3$
bad polys
$Z(b^2-4ac)$

Application: Complements of varieties are varieties

e.g. $\text{Poly}_2(\mathbb{C}) = \{ax^2+bx+c : b^2-4ac \neq 0\}$
$GL_2(\mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc \neq 0\right\}$

③ Grassmannian

$G_{r,n} = \{$ r-dim planes thru 0 in $k^n \}$

Note: $\mathbb{P}^n = G_{1,n+1}$

$G_{r,n}$ important in topology:
"classifying space for n-dim vector bundles"

We'll show this a proj. var

Plücker embedding:

$$G_{r,n} \longrightarrow \mathbb{P}(\Lambda^r k^n)$$

"parameter space of widgets is a widget"

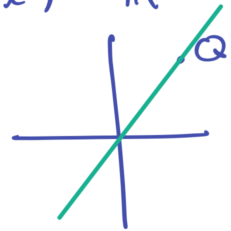
④ Blow up

(Fixing, not destroying)

The map $\mathbb{A}^2 \setminus 0 \longrightarrow \mathbb{P}^1$ does not extend to 0.

The blowup of $\mathbb{A}^2$ at 0:

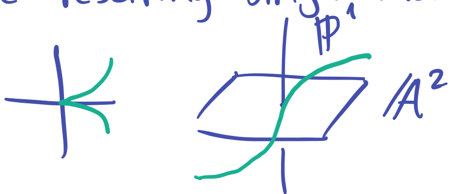
$$\{(Q, \ell) \in \mathbb{A}^2 \times \mathbb{P}^1 : Q \in \ell\}$$



get a copy of $\mathbb{P}^1$ at 0.

Application: resolving singularities.

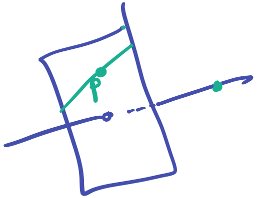
$x^3 = y^2$



Chapter 4. Dimension, degree, smoothness

"expected properties"

Dim $\dim_p V$ = (size of max chain of varieties at p) − 1



We'll show: behaves like dim in lin alg.

codim $V_i = c_i$ (V_i irred)

codim $V_1 \cap V_2 = c_1 + c_2$

generically.

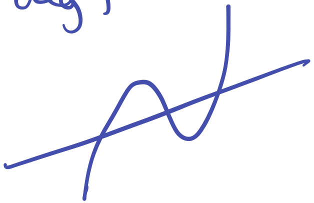
Degree $V \subseteq \mathbb{A}^n$ or $\mathbb{P}^n$ k alg closed

$\dim V = k$.

$\deg V$ = generic/expected # intersections with $n-k$ plane

For $V = Z(f)$ "hypersurface"

$\deg V = \deg f$



Helps understand number of solns when $\dim = 0$.

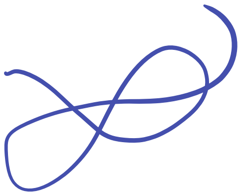
Smoothness A $\mathbb{C}$-variety is <u>smooth</u> exactly when it is a manifold... can use manifold theory.

Chapter 5. Curves in the plane.

Setup $f \in \mathbb{C}[x_0, x_1, x_2]$ homog.

$C = Z(f) \subseteq \mathbb{P}^2$

Picture:



When $\deg f = 2$

C is a "conic"

Thm. (Five pts determine a conic)

Given $p_1, \dots, p_5 \in \mathbb{P}^2$ $\exists$ conic passing thru all p_i (generically unique)

Bézout's thm. $C_1 = Z(f_1)$ $\deg f_i = d_i$

$C_2 = Z(f_2)$

& f_1, f_2 no common factor

Then $|Z(f_1) \cap Z(f_2)| = d_1 d_2$

(count with mult.)

Thm. Given $\binom{d+1}{2}$ distinct pts in $\mathbb{P}^2$

$\exists$ deg d curve passing thru.

Cayley-Bacharach Thm

$C_1, C_2 \subseteq \mathbb{P}^2$ cubic curves

with $|C_1 \cap C_2| = 9$

If C_3 (cubic curve.) passes thru 8 of the pts,

it passes thru the 9^{th}.

Smoothness for curves:

$$C = Z(f)$$

smooth a p if some

$\frac{df}{dx_i}(p)$ nonzero.

Thm. $f \in \mathbb{C}[x_0, x_1, x_2)$

irred, homog, deg d

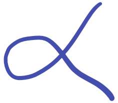
$\rightsquigarrow Z(f)$

Then # sing pts $\leq \binom{d-1}{2}$

Classification of cubic curves in $\mathbb{P}^2$

1 sing pt: equiv to

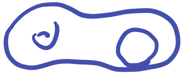
$Z(x^2 = y^3) \subseteq \mathbb{A}^2$



$Z(y^2 = x^2 + x^3)$

$\subseteq \mathbb{A}^2$

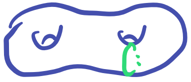
Actual pic



Smooth:

$Z(y^2 = 4x^3 - g_2 x - g_3)$

"Weierstrass curves"



Chapter 6. Special topics

Cayley-Salmon thm

Every smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ contains 27 lines.

e.g. $x^3+y^3+z^3+w^3$

find the lines!