Chapter 1. Affine alg. vars & the geometry/alg dictionary

Setup: $S \subseteq k[x_1, \dots, x_n]$

$\rightsquigarrow Z(S) = \{a \in \mathbb{A}^n : f(a) = 0 \ \forall f \in S\}$

"affine alg var"

Examples

① $\emptyset = Z(k[x_1, \dots, x_n])$
$= Z(1)$

Second = Makes sense since $(1) = k[x_1, \dots, x_n]$

② $\mathbb{A}^n = Z(0)$

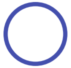
③ $(a_1, \dots, a_n) = Z(x_1 - a_1, \dots, x_n - a_n)$

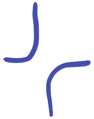
compare: lin alg.

④ (Hyper)planes

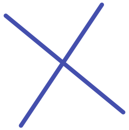
⑤ Conics $Z(f) \subseteq \mathbb{A}^2$ $\deg f = 2$.

e.g. $x^2 - y^2 - 1$ $xy - 1$ $y - x^2$





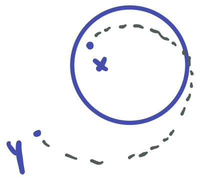
$(x-y)(x+y)$ $x^2 - 1$ x^2



"double line"

Aside: Conics over $\mathbb{C}$ from a topological pt of view.

① $Z(xy-1)$ is connected over $\mathbb{C}$

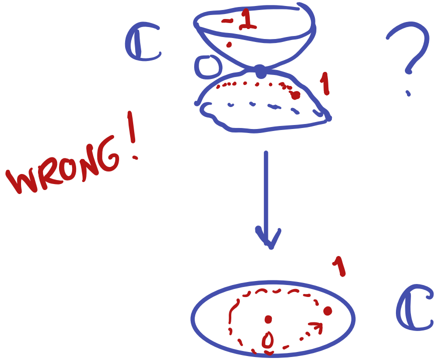


every pt connected to $(1,1)$

② $Z(x^2-y)$ over $\mathbb{C}$.

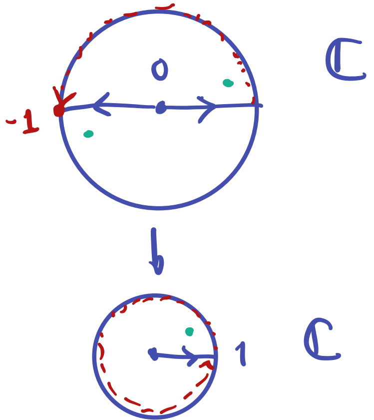
Have a map

$$Z(x^2-y) \longrightarrow \mathbb{C}$$
$$(x,y) \longmapsto y$$



? 2 preimages (away from 0)

Right picture



"Riemann surface"

⑥ Algebraic groups

$SL_n k = Z(\det - 1) \subseteq \mathbb{A}^{n^2}$

$GL_n k$ complement of $Z(\det)$ by defn.

In general, complements of aar's are aav's (later)

To see $GL_n k$ as a variety:

$V = \{(x_{ij}, t) \in \mathbb{A}^{n^2+1} : \det(x_{ij})t - 1 = 0\}$

$\varphi : GL_n k \longrightarrow V$

$A = (a_{ij}) \longmapsto \left(a_{ij}, \frac{1}{\det A}\right)$

is a bijection.

⑦ Twisted cubic

$C = \operatorname{Im} \varphi$ where

$\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^3$

$t \longmapsto (t, t^2, t^3)$

As a variety

$C = Z(x^2 - y, x^3 - z)$

$= Z(x^2 - y, z - xy)$

intersection of two "quadrics"

C is also a determinantal var

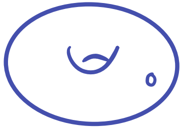
$C = \{(x, y, z) \in \mathbb{A}^3 : \operatorname{rank}\begin{pmatrix} 1 & x & y \\ x & y & z \end{pmatrix} < 2\}$

(Chris) Q. Is any int. of quadrics a det. var?

⑧ A family of (smooth) cubics

$$C_\lambda = Z(x(x-1)(x-\lambda) - y^2) \subseteq \mathbb{A}^2$$

$\lambda \neq 0, 1 \qquad k = \mathbb{C}$

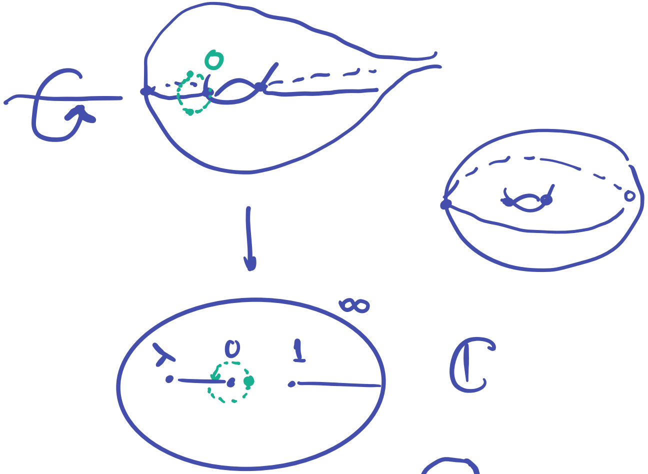
Claim: $C_\lambda \cong$ 

Like the $x - y^2$ example:

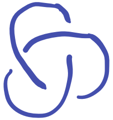
$$C_\lambda \longrightarrow \mathbb{A}^1$$
$$(x,y) \longrightarrow x$$

Other than $x = 0, 1, \lambda$

pts in $\mathbb{A}^1$ have two preims.



⑨ Trefoil



$$Z((x^2+y^2)^2 + 3x^2y - y^3)$$

intersect with $S^3 = \{(x,y) : |x^2| + |y^2| = 1\}$

"singularity theory"

exercise: Take complement of axes in $\mathbb{C}^2$

& intersect with S^3 ....

Nonexamples $k = \mathbb{C}$

① **Fact.** Every aff alg var (in $\mathbb{A}^n$) is closed in Euclidean top

$\Longrightarrow \{z : |z| < 1\}$ not an aav in $\mathbb{A}^n$

② **Fact.** The interior of any proper aav is $\emptyset$.

$\longrightarrow \{z : |z| \leq 1\}$ not an aav.

③ **Fact.** Any proper variety in $\mathbb{A}^1$ is finite (by FTAlg)

$\Longrightarrow \mathbb{Z} \subseteq \mathbb{C}$ is not a.a.v.

Basic Properties of aav's

① $\forall\, S \subseteq k[x_1, \ldots, x_n]$ have

$$Z((S)) = Z(S)$$

(exercise)

② Intersections of aav's are aav

$$\bigcap_\alpha Z(I_\alpha) = Z\left(\bigcup I_\alpha\right)$$

③ Finite unions of aav's are aav's

$$V(I) \cup V(J) = V(IJ)$$

(exercise) ↑ finite sums $\sum i_k j_k$

example $V(x) \cup V(y) = V(xy)$

Zariski Topology

A topology on a space X is a collection of sets, called closed sets such that

1. $\emptyset$, X closed
2. Finite unions of closed sets are closed
3. Arbitray intersections of closed sets are closed.

Complements of closed sets called "open".

Def. Zariski topology on $\mathbb{A}^n$ has aav's as the closed sets.

Basic properties $\Rightarrow$ this indeed is a topology.

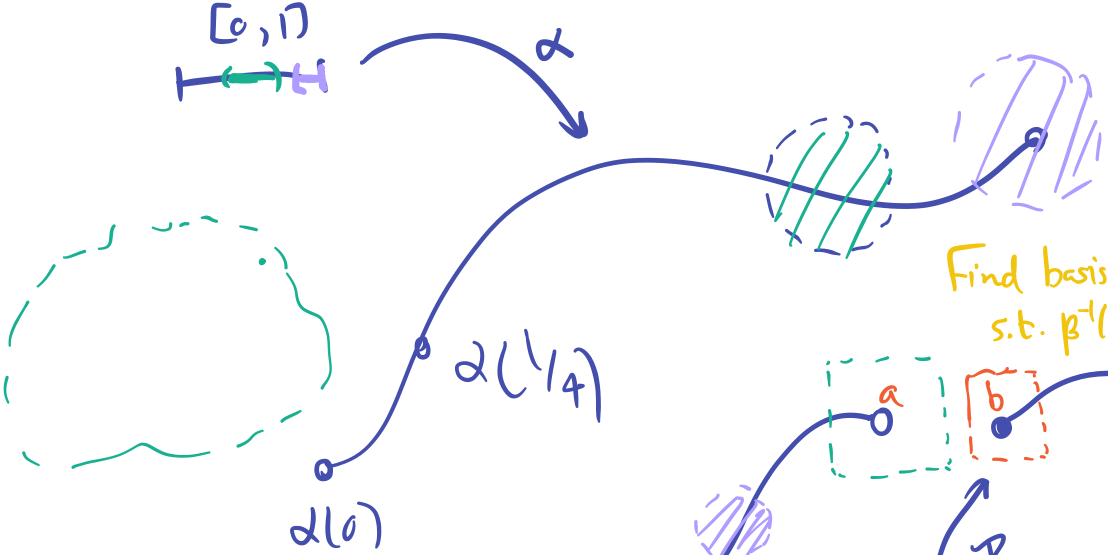
The Zariski top. is strange:

1. All proper closed sets have $\emptyset$ interior.
2. Proper closed subsets of $\mathbb{A}^1$ are finite.
3. No two open sets are disjoint.
   $\Rightarrow$ not Hausdorff.
4. compact $\not\Rightarrow$ closed
   closed $\Rightarrow$ compact

A concept: A set is Zariski dense iff every polynomial is det. by its values on that set.

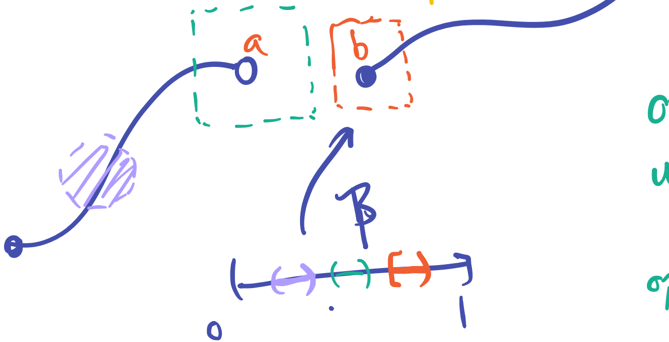
e.g. $\mathbb{Z} \subseteq \mathbb{A}^1_{\mathbb{R}}$

ε-δ defn $\Longleftrightarrow$ ~~open~~ basis set defn

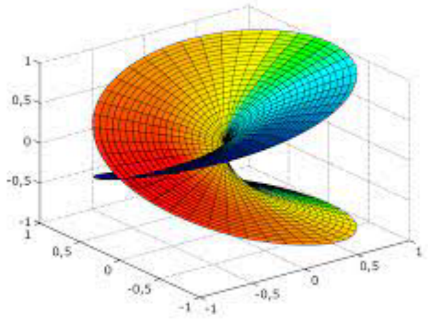


open set in $\mathbb{R}$ is a union of (a,b)

Find basis elt U of $\mathbb{R}^2$ s.t. $\beta^{-1}(U)$ not open



open set in $\mathbb{R}^2$: union of product of open intervals


1
0,5
0
-0,5
-1
1
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0
-0,5
-1
-1
-0,5
0
0,5
1