Smooth cubic curves

Hulek

Last time: every smooth irred cubic in $\mathbb{P}^2$ is proj. equiv to

$C_{b,c} = Z(f_{b,c})$

$f_{b,c} = y^2 - 4x^3 + bx + c$.

Also: $C_{b,c}$ smooth $\Leftrightarrow \mathrm{Disc}(f_{a,b}) \neq 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad b^3 - 27c^2$

Conseq. $\{\text{Smooth } C_{b,c}\}$ is $\cong$ a.a.v.

J-invt

$J: \{C_{b,c}\}^{\text{smooth}} \longrightarrow \mathbb{C}$

$C_{b,c} \longmapsto \dfrac{b^3}{b^3 - 27c^2}$

Equiv. reln on $\{C_{b,c}\}$: differ by proj aut fixing $[0:0:1]$.

4 coord

Prop. $C_{b,c} \sim C_{b',c'}$ (smooth) $\Leftrightarrow$ same J

Lemma. Any proj aut. fixing $[0:0:1]$ is of form

$x \longmapsto u^2 x$

$y \longmapsto u^3 y$

Pf. lin alg...

Prop. $C_{b,c} \sim C_{b',c'} \iff$ same j
(smooth)

Pf. Special case $j=0$

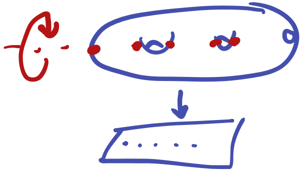
$\Rightarrow$ easy using lemma.

$\Leftarrow$ $j=0 \Rightarrow b=b'=0,\ c\neq 0.$

Choose u s.t. $c' = c/u^6$ $\square$

By Prop:

$$j : \{\text{smooth } C_{b,c}\}/\sim \ \longleftrightarrow \ \mathbb{C}.$$



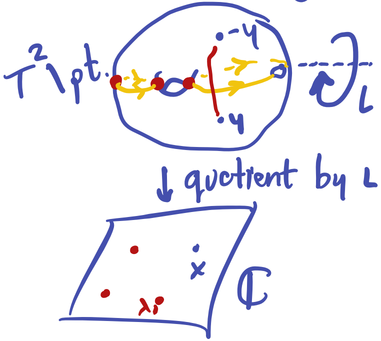
Another point of view $k=\mathbb{C}$.

Every $C_{b,c}$ is homeo to $\circledcirc = T^2$

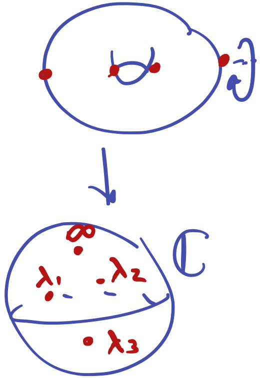
$$y^2 = 4x^3 - bx - c = 4(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

$C_{b,c}$ has an involution $(y,x) \to (-y,x)$

In $\mathbb{A}^2$



project'ivize

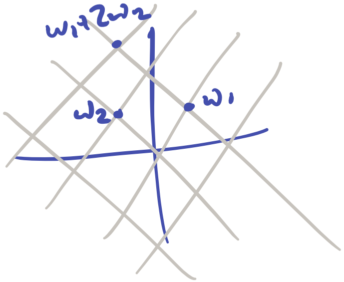


Upshot: $C_{b,c} \cong T^2$ as Riemann surf. (complex manifold)

Another way to make a torus

$\omega_1, \omega_2 \in \mathbb{C} \rightsquigarrow$

$$\Lambda = \{\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2\}$$



$E_\Lambda = \mathbb{C}/\Lambda \cong T^2$ "elliptic curve"

equiv: biholomorphism.

Will show: $\{E_\Lambda\}/\sim \;\Longleftrightarrow\; \left\{\begin{matrix}\text{smooth}\\ C_{b,c}\end{matrix}\right\}/\sim$
$\updownarrow$ $\mathbb{C}$

Equivalence on $\{E_\Lambda\}$.

Given Λ, can rotate, ~~flip~~ negate, scale so

$\omega_1 = 1$

$\operatorname{Im} \omega_2 > 0$ $\quad(\omega_1 = 1, \ \omega_2 = \tau)$
$\tau \in$ upper half plane

$E_\Lambda \cong E_\tau$

Moreover: $SL_2\mathbb{Z} \circlearrowright$ upper half-plane by Möbius transf.

__Fact__. $E_\tau \sim E_{\tau'} \iff \tau \sim \tau' \mod SL_2\mathbb{Z}$.

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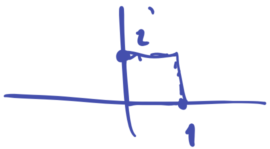
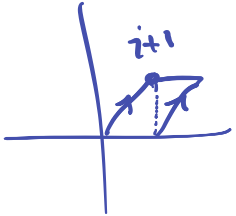
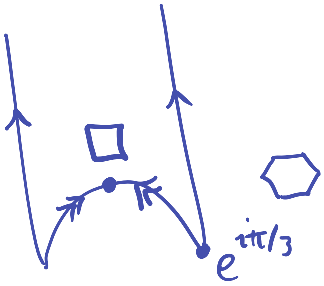
Example $\tau = i$.



$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot i = \frac{1\cdot i+1}{0\cdot i+1} = i+1 = \tau'$$

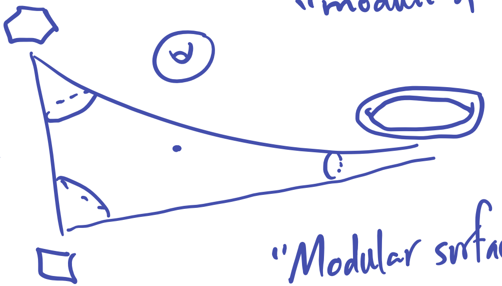


Fund dom for $SL_2\mathbb{Z} \curvearrowright \mathbb{H}$ upper half plane



"moduli space"

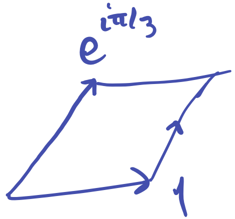
Quotient $\{E_\Lambda\}/\sim$ =



"Modular surface"

Note: Mod. surf homeo to $\mathbb{C}$

Hexagonal torus



Cut & paste into a reg. hexagon

Now have: $\left\{\begin{matrix}\text{smooth}\\ C_{b,c}\end{matrix}\right\}/\sim$ & $\underset{\text{mod. surf}}{\overset{\shortparallel}{\{E_\Lambda\}/\sim}}$

both homeo to $\mathbb{C}$.

Want: Map between them.

<u>Weierstrass $\wp$ function</u>

Assume $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$

$$\wp(z) = \wp_\Lambda(z) = $$

$$= \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus 0} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

Invariant under Λ, i.e. it is a fn on E_Λ

Get a map:

$$\varphi : E_\Lambda \longrightarrow C_{b,c}$$

$$z \longmapsto [1 : \wp(z) : \wp'(z)]$$

y

where

$$b = 60 \sum_{\omega \in \Lambda \setminus 0} \frac{1}{\omega^4}$$

$$c = 140 \sum \frac{1}{\omega^6}$$

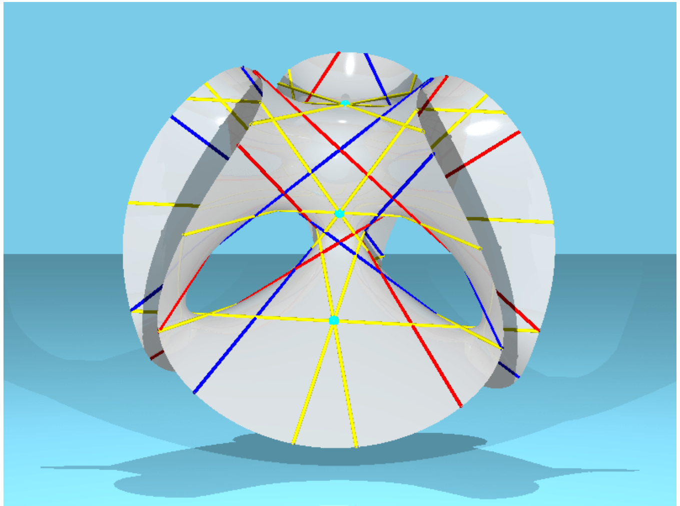
Works because

$$(\wp')^2 = 4\wp^3 - b\wp - c.$$

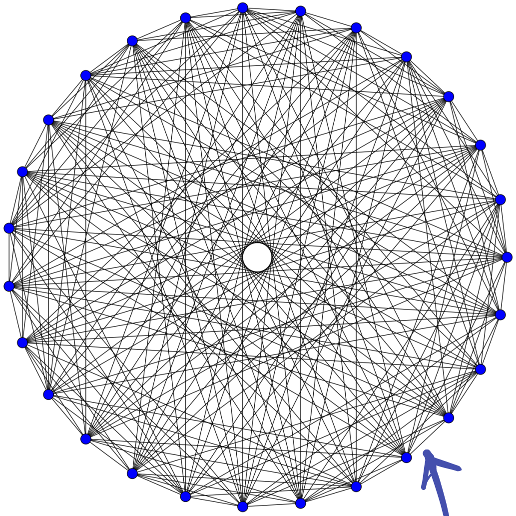
This the desired map $\underset{\text{mod. surf.}}{\overset{\shortparallel}{\{E_\Lambda\}/\sim}} \to \left\{\begin{matrix}\text{Smooth}\\ C_{b,c}\end{matrix}\right\}$

Injectivity: J-invt.

Surj: J is holom. nonconst. map



Clebsch



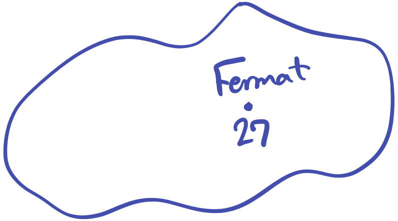
Cayley - Salamon Thm: Every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.

and the (non)-intersection pattern given by

Basic strategy

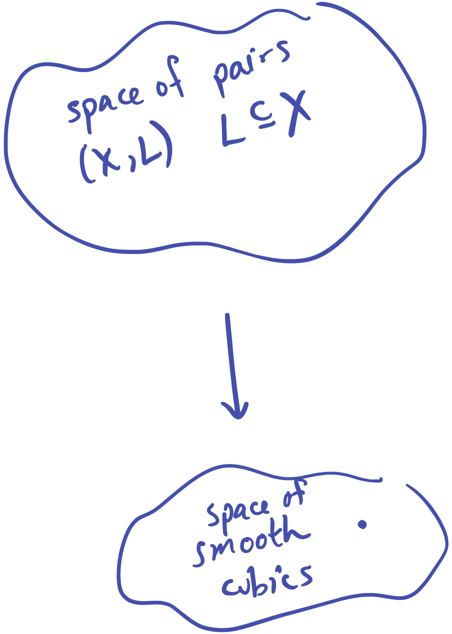
① Show that $Z(x^3+y^3+z^3+w^3)$ "Fermat curve" has exactly 27 lines.

② The # of lines is locally const. in moduli space of smooth cubic surfaces (which is connected)



smooth cubic surfaces

In alg. top. language.



deg 27 cov. space.