27 LINES

A cubic surf. is $S = Z(f) \subseteq \mathbb{P}^3$ where $\deg f = 3$.

Gathmann

Cayley-Salamon Thm

S smooth $\Longrightarrow$ S contains exactly 27 lines

Strategy. Show ① that some S has 27 lines and ② # lines is locally const. in space of smooth cubic surfaces.

The some S is Fermat cubic:

$$Z(x_0^3 + x_1^3 + x_2^3 + x_3^3)$$

Lemma. The Fermat cubic X has 27 lines (exactly)

$$X = Z(x_0^3 + x_1^3 + x_2^3 + x_3^3)$$

Pf. X invt under permutation of coords.

Up to such permut, any line is

$$x_0 = a_2 x_2 + a_3 x_3$$
$$x_1 = b_2 x_2 + b_3 x_3$$

(move the 2 pivots to left)

Such a line lies in $X \iff$

$$0 = (a_2x_2 + a_3x_3)^3 + (b_2x_2 + b_3x_3)^3 + x_2^3 + x_3^3$$

Compare coeffs of LHS = 0 & RHS

$\rightsquigarrow a_2^3 + b_2^3 = -1$ (1) x_2^3 term

$a_3^3 + b_3^3 = -1$ (2)

$a_2^2 a_3 = -b_2^2 b_3$ (3)

$a_2 a_3^2 = -b_2 b_3^2$ (4)

If a_2, b_2, a_3, b_3 all $\neq 0$ then $(3)^2/(4)$

$\rightsquigarrow a_2^3 = -b_2^3$ contradicting (1).

So WLOG $a_2 = 0$.

(1) $\Rightarrow b_2^3 = -1$

(3) $\Rightarrow b_3 = 0$

(2) $\Rightarrow a_3^3 = -1$

permute coords → other 18.

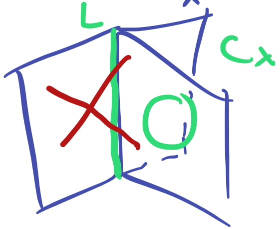
$\rightsquigarrow$ 9 lines (3 choices for each $\sqrt[3]{-1}$) $\square$

How are the lines related?

Intersection pattern: (complement of) Schläfli graph

Claim: Each of the 27 lines in a cubic surface intersects 10 of the others.

Idea: Given one line L, consider the family of planes $\{P_\lambda\}_\lambda \subseteq \mathbb{P}^3$, $\lambda \in \mathbb{P}^1$, containing L.



$P_\lambda \cap S$ = cubic curve X_λ

By our classification:

X_λ = 3 lines $L \cup L' \cup L''$ (C_λ)

or $L \cup C_\lambda$ conic

(need to rule out double lines).

C_λ being 2 lines or conic is a smoothness/Jacobian condition

$\rightsquigarrow$ deg 5 poly in λ. (discrim).

For each of the 5 roots, get 2 lines intersecting L.

Moduli space of smooth cubic surfaces

All cubic surfaces:

$$\mathbb{P}^{19} = \mathbb{P}^{\binom{3+3}{3}-1}$$

3 balls in 4 boxes

Claim:

Smoothness for $Z(f)$

$\iff$ rk $(df/dx_i) \neq 0$.

($\iff$ rk ≥ 1

$\iff$ tangent space ≤ 2.)

Lucky accident: The zeros of $\frac{df}{dx_i}$ are all on $Z(f)$

Why? Euler's identity

$$3f = \sum x_i \frac{df}{dx_i}$$

By claim: Moduli space of smooth cubic surfaces is complement of intersection of 4 hypersurfs in $\mathbb{P}^{19}$.

$\rightsquigarrow$ dense & open in $\mathbb{P}^{19}$.

$\Rightarrow$ moduli sp. is connected (codim reasons)

The Incidence Correspondence

U = mod. sp. of sm. cub. surf

$\mathbb{G}_{1,3}$ $\shortparallel$

$M = \{(X, L) : L \subseteq X\} \subseteq U \times Gr_{2,4}$

There is projection $\pi : M \to U$.

$(X, L) \to X$

Have:

$\#$ lines in X $= |\pi^{-1}(X)|$

WTS this $\#$ is indep of X.

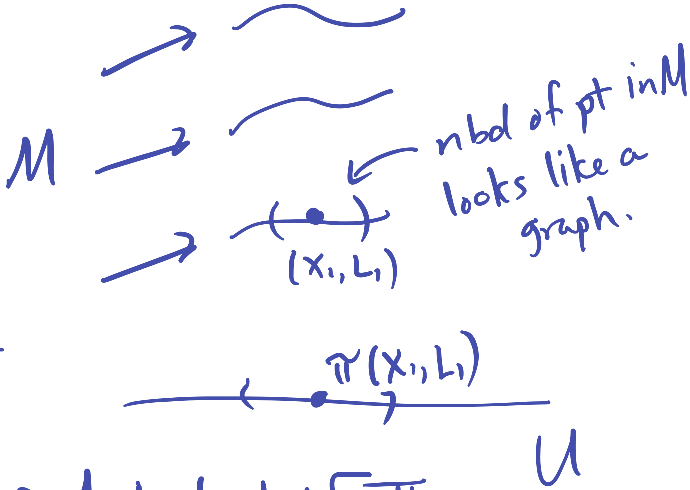
Two ways of rephrasing this:

① $M \to U$ is a cov. sp.

② $|\pi^{-1}(x)|$ is locally const.

Main idea M is locally* the graph of a continuously diff. fn $U \to Gr_{2,4}$.

* in classical topology.



Pf. Apply Implicit Fn Thm. □

Blowing up $\mathbb{P}^2$

Thm. Every smooth cubic surface S is the blowup of $\mathbb{P}^2$ at 6 pts.

Cor. $S \overset{\text{homeo}}{\cong} \mathbb{CP}^2 \#_6 \overline{\mathbb{CP}^2}$

$\Rightarrow \pi_1(S) = 1$

$\Rightarrow H_2(S) \cong \mathbb{Z}^7$

(intersection form type $(1,-6)$...)

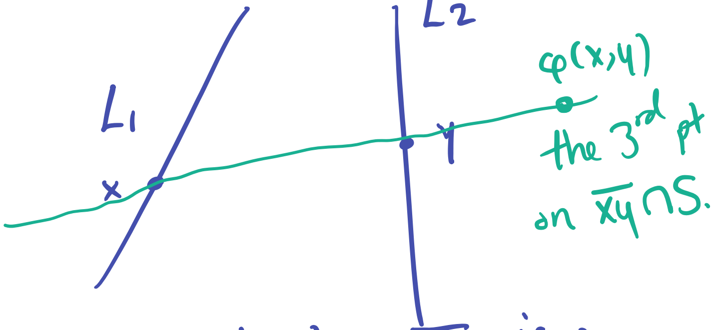
Idea of Thm

Further analysis of above work

$\Longrightarrow$ S has 2 disjoint lines L_1, L_2.

(we found the ones that intersect)

Define map $\varphi : L_1 \times L_2 \dashrightarrow S$



Works except when $\overline{xy}$ is one of the 27 lines.

Need to blow up $L_1 \times L_2$
in 5 pts to get well def map.

And: $L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$

$\mathbb{P}^1 \times \mathbb{P}^1$

$\cong$ $\mathbb{P}^2$ blown up at 1 pt.

(stereographic proj.