Last time:

$$\dim X = \dim k[X]$$
$$= d \text{ s.t. } \exists \text{ finite}$$
$$X \longrightarrow \mathbb{P}^d.$$

Hulek Chap 3.

Today:

$$\dim X = \min_{p \in X} \dim T_p X$$
$$= \operatorname{tr\,deg}_k k(X)$$

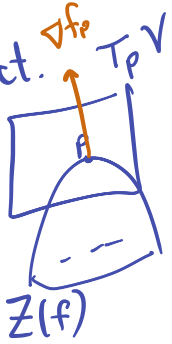
Tangent spaces

Note: f is well-def. object when X irred.

$X = Z(f) \subseteq \mathbb{A}^n$ irred. hypersurf.

$$\nabla f_p = \left(\frac{df}{dx_1}(p), \ldots, \frac{df}{dx_n}(p) \right) \in k^n.$$

$\rightsquigarrow \nabla f_p \in (k^n)^*$ via dot product.



$$T_p X = p + \ker \nabla f_p$$

Write

$$f_p^{(1)} = \sum \left(\frac{df}{dx_i}(p) \right) (x_i - p_i)$$

"linear part of f at p"

$T_p X$ is the set of solns.

Examples

(0) Hyperplane $H \subseteq \mathbb{A}^n$

$T_pH = H$ (exercise).

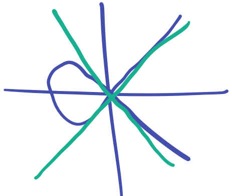
(1) Parabola $f(x,y) = y - x^2$

$\rightsquigarrow \nabla f = (-2x, 1)$

$\rightsquigarrow \nabla f_0 = (0, 1)$

$\rightsquigarrow T_0 X = x\text{-axis}.$

(2) $X = Z(y^2 - x^2 - x^3)$

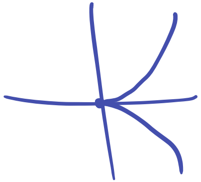


$\nabla f = (-2x - 3x^2, 2y)$

$\nabla f_0 = (0, 0)$

$\rightsquigarrow T_0 X = \mathbb{A}^2.$

(3) $X = Z(y^2 - x^3)$



$\rightsquigarrow T_0 X = \mathbb{A}^2$

(4) $X = Z(x^m - y^m)$

Check: $T_{(a,b)} X$ is a line if $\operatorname{char} k \nmid m$.

not irred — so we need more defns!

Projective varieties

To define T_pX, pass to affine chart, take tang. sp there, take proj. closure.

Tangent spaces & roots

Prop. $L \subseteq \mathbb{A}^n$ affine line.

$p \in L$

$X = Z(f) \subseteq \mathbb{A}^n$

Then $L \subseteq T_pX \iff$

$f|_L$ has a multiple root at p.

examples. ① $X = Z(y - x^2)$

$x^2 = 0$ ✓

② $X = Z(y^2 - x^3)$.

$L : y = tx$

$\rightsquigarrow (tx)^2 - x^3 = x^2(t^2 - x)$ ✓

mult. root at 0

Pf. Let $L(t) = (p_1 + b_1 t, \dots, p_n + b_n t)$

Let $g(t) = f|_L = f(p_1 + b_1 t, \dots, p_n + b_n t)$

Know $g(0) = f(p) = 0$.

Want $g'(0) = 0$. By chain rule:

$\frac{dg}{dt}(0) = 0 \iff \sum b_i \frac{df}{dx_i}(p) = 0$

$\iff L \subseteq T_pV$ $\square$

Tangent sp for general irred (not just hypersurf)

$X \subseteq \mathbb{A}^n \qquad X = Z(f_1, \dots, f_m)$

$T_p X = \bigcap_{f \in \mathbb{I}(X)} T_p Z(f)$

$\overset{\text{exercise}}{=} \bigcap_{i=1}^{m} T_p Z(f_i)$

Smoothness

$X = Z(f) \subseteq \mathbb{A}^n$ irred hypersurf.

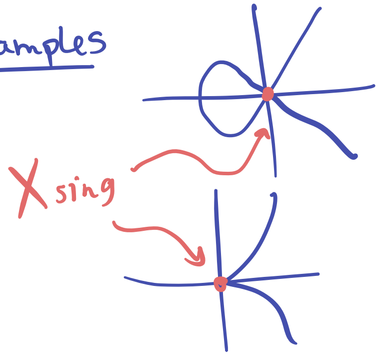
$p \in X$ smooth if $\nabla f_p \neq 0. \iff T_p X \cong \mathbb{A}^{n-1}$

singular o.w. $\iff T_p X = \mathbb{A}^n$

$\rightsquigarrow X_{smooth}$

$X_{sing} = X \setminus X_{smooth}.$

examples



Prop. $X = Z(f) \subsetneq \mathbb{A}^n$ irred.
$X_{smooth} \subseteq X$ open, dense

Pf (char $k = 0$)

To show: ① X_{sing} closed
② $X_{smooth} \neq \emptyset$.

① $X_{sing} = Z\left(f, \frac{df}{dx_1}, \ldots, \frac{df}{x_n}\right)$

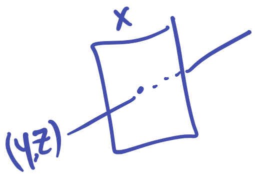
② Assume $X_{sing} = X$.
$\Longrightarrow \frac{df}{dx_i} \in (f) \quad \forall i.$ ← since f irred.

Since f not const, this is a contrad. (look at degrees)

Note over $\mathbb{C}$, X smooth is a complex manifold (inverse fn thm).

The reducible case

If X has irred. comp's $\{X_i\}$ say p is smooth if it lies in exactly one X_i & is smooth as a pt in X_i.



$X = Z(xy, xz)$

O is smooth in both components.
but not smooth by our defn.

Back to dimension

Let's write

$$\dim X = \min_{p \in X} \dim T_p X$$

for X irred.

If X is red. with irred comp's X_i,

$$\dim X = \max \dim X_i .$$

Above examples ① - ④ have dim 1.

Prop. $X = Z(f) \subsetneq \mathbb{A}^n$ irred. hypersurf.

$$\Rightarrow \dim X = n-1 .$$

Prop. $X \subseteq \mathbb{A}^n$ irred.

$\exists$ open, dense $X_0 \subseteq X$ s.t.

$$\dim T_p X = \dim X \quad \forall p \in X_0 .$$

Lemma. $X \subseteq \mathbb{A}^n$ irred. $\forall r \in \mathbb{N}$. The set $S_r(X) = \{p \in X : \dim T_p X \geq r\}$ is closed

Pf. Say $\mathbb{I}(X) = (f_1, \ldots, f_m)$

$$T_p X = \bigcap Z((f_i)_p^{(1)})$$

$$\Rightarrow \dim T_p X = n - \operatorname{rank}\left(\frac{df_i}{dx_j}(p)\right) \quad \square$$

Pf of Prop. Let $r = \dim X \rightsquigarrow S_r X = X$, $S_{r+1} X \neq X$. $\square$

set of pts with wrong dim

Back to smoothness

X irred, maybe not hypersurf.

$X = Z(f_1, \dots, f_m)$

$p \in X$ is **smooth** if

$$\operatorname{rank}\left(\frac{df_i}{dx_j}\right) = m.$$

Fact.
p smooth $\iff \dim T_p X = \dim X$

Codim.

$X = Z(f_1, \dots, f_m)$ irred. $\subseteq \mathbb{A}^n$

$$\operatorname{codim} X = n - \dim X$$
$$= \operatorname{rank}\left(\frac{df_i}{dx_j}(p)\right) \quad p \text{ smooth.}$$

$\Longrightarrow \operatorname{codim} X \leq m.$

(also true for red.)

Alg char of dim

We'll show

$$\dim X = \operatorname{trdeg}_k k(X) \overset{\text{algebra}}{=} \dim k[X]$$

Hypersurface case

$$k[X] = k[x_1, \dots, x_n]/(f)$$

WLOG f uses x_1

$$k(X) = k(x_2, \dots, x_n)[x_1]/(f)$$

which clearly(?) has transc. deg $n-1$.