Goal for today:

coord free description of tangent spaces.

We'll show

$$T_pX \cong (M/M^2)^* \cong (m/m^2)^*$$

$$M \subseteq k[x_1,\dots,x_n]$$

$$m \subseteq k(x_1,\dots,x_n)$$

max ideals at p:

$$M = (x_1-p_1,\dots,x_n-p_n)$$

fns that vanish at p

Cotangent spaces

$$T_p^*V = \text{dual of } T_pV$$
$$= \{\text{linear } T_pV \to k\}$$
$$= \text{"linear forms"}$$

Notation: $g \in k[x_1,\dots,x_n]$

d_pg = diff. of g at p.

e.g. $g(x,y) = x^2 + xy + x$

$$dg = (2x+y+1)\frac{d}{dx} + x\frac{d}{dy}$$

$$d_0g = \frac{d}{dx}. \in (k^n)^*$$

Prop. Let $X = Z(f_1,\dots,f_m) \subseteq \mathbb{A}^n$

$p \in X \quad g \in k[X]$

$\rightsquigarrow d_p g$ is lin form on $T_p X$.

Justin R. Smith

Pf. To show well-def.

Say $G_1, G_2 \in k[x_1,\dots,x_n]$ rep. g.

$\Rightarrow G_1 - G_2 = \sum h_i f_i \quad h_i \in \mathbb{I}(X)$

$\Rightarrow d_p(G_1 - G_2) = \sum\left((d_p h_i) f_i(p) + h_i(p)(d_p f_i)\right)$

product rule.

0 by defn

$T_p V$ defined so that this form evals to 0 on it.

$= 0$ $\square$

Prop. Same X. Differentiation induces surj $M \longrightarrow T_p^* V$ with kernel M^2.

Pf. Setup. WLOG $p = 0$.

WLOG $T_p V = \langle x_1, \dots, x_r \rangle$ (change of coords)

Let $\tilde{M} = (x_1, \dots, x_n) \subseteq k[x_1,\dots,x_n]$

Its image in $k[X]$ is M.

Prop. Same X. Differentiation induces surj $M \longrightarrow T_p^*X$ with kernel M^2. So: $T_p^*X = M/M^2$.

Pf. Setup. WLOG $p=0$.

WLOG $T_pX = \langle x_1, \dots, x_r \rangle$ (change of coords)

Let $\tilde{M} = (x_1, \dots, x_n)$

Its image in $k[X]$ is M.

Surjectivity: let $\ell = \sum c_i x_i^* \in T_p^*X$

Then $L = \sum c_i x_i$ has $dL = \ell$

Kernel: Say $g \in M$, $d_0 g \equiv 0 \in T_p^*X$ & g is image of $G \in \tilde{M}$

So $d_0 G \equiv 0$ on T_0X (first Prop)

Then $d_0 G = \sum \lambda_j (d_0 f_j)$ (by defn of T_pX)

Let $\bar{G} = G - \sum \lambda_j f_j$

Then $\bar{G}$ still maps to g in $k[X]$.

But $d_0 \bar{G} \equiv 0$ on $T_0 \mathbb{A}^n$

$\Longrightarrow$ const & lin. terms of $\bar{G}$ vanish.

$\Longrightarrow \bar{G} \in \tilde{M}^2 \Longrightarrow g \in M^2$ $\square$

Moving the *

R = ring, $M \subseteq R$ max ideal.

$\rightsquigarrow R \cdot M \subseteq M$, $R \cdot M^2 \subseteq M^2$

So M, M/M^2 modules over R

Also, mult by M on M/M^2 is 0 map.

So M/M^2 is R/M-module
$\quad\quad\quad\quad\quad\quad\quad\quad$ ↳ field!

i.e. M/M^2 is vect sp. over R/M

So $T_pV = (M/M^2)^*$ makes sense

$(M/M^2)^*$ is called Zariski tangent sp.

Differentials

Prop. $f: X \to Y$ morphism of aav's
$p \in X$

$\rightsquigarrow f_* : T_pX \to T_{f(p)}Y$

Pf. $f_* : k[Y] \to k[X]$

prein of M is N = max ideal for $f(p)$
= fns vanish @ $f(p)$

So $N/N^2 \longrightarrow M/M^2$ $\square$

Coord free descr. of differential

Prop. $X \subseteq \mathbb{A}^n$ irred.

$f \in k[X]$

Then $f - f(p) \in M$.

and $d_p f$ = image of $f - f(p)$

in $M/M^2 = T_p^* V$

Pf. Subtracting $f(p)$ kills const term.

Modding by M^2 kills quad & higher terms.

$\square$

you!

Example $X = Z(x^3 - y^2) \subseteq \mathbb{A}^2$

At $p = (1,1)$ can see $\dim M/M^2 = 1$:

$M = (x-1, y-1)$

$\rightsquigarrow M^2 = (x^2 - 2x + 1, (x-1)(y-1), \cancel{y^2}^{\,x^3} - 2y + 1)$

$\rightsquigarrow y - 1 = (x^3+1)/2 - 1$

$= (x(2x-1)+1)/2 - 1$

$= (2x^2 - x + 1)/2 - 1$

$= (3x-1)/2 - 1$

$= 3/2\,(x-1)$

At $p = (0,0)$ can see $\dim M/M^2 = 2$:

$M = (x, y)$, $M^2 = (x^2, xy, y^2)$

$\rightsquigarrow M/M^2 = \{ax + by\}$

Projective varieties

$\mathcal{O}_{X,p} = \{ f/g \in K(X) : g(p) \neq 0 \}$

$\mathfrak{m} \in$ $f/g \in \mathcal{O}_{X,p}$ s.t. $f(p) = 0$

max ideal.

Lemma. $X, M, \mathfrak{m}, p$ as above.

$M/M^2 \cong \mathfrak{m}/\mathfrak{m}^2$

Pf. WLOG $p = 0$.

Inclusion $M \hookrightarrow \mathfrak{m}$

Induces injection

$$M/M^2 \hookrightarrow \mathfrak{m}/\mathfrak{m}^2$$

Surj Let $f/g \in \mathfrak{m}/\mathfrak{m}^2$ so $g(0) \neq 0$

$$\rightsquigarrow f/g(0) - f/g$$

$$= f\left(1/g(0) - 1/g \right) \in \mathfrak{m}^2$$

So $f/g(0) = f/g$ in $\mathfrak{m}/\mathfrak{m}^2$

in $M \subseteq k[X]$

$\square$

Cor 1. $f : X \dashrightarrow Y$ rat.

$\rightsquigarrow f_* : T_pX \to T_{f(p)}Y$.

Cor 2. X, Y birat $\Rightarrow \dim X = \dim Y$

Back to dim

Thm. $X \subseteq \mathbb{A}^n$ irred

$$\dim X = \operatorname{tr\,deg}_k k(X).$$ Milne

Also: $\operatorname{trdeg}_k k(X) = \dim k[X]$.

Pf. If it's true for hypersurfaces, true for all X since every X is birat. equiv to hypersurf. (Noether norm). Hulek

For hypersurfaces:

We proved $\dim = n-1$ so suff. to show $\operatorname{tr\,deg} = n-1$

$X = Z(f) \subseteq \mathbb{A}^n$ f irred.

$\rightsquigarrow k[X] = k[x_1, \dots, x_n]/(f)$

WLOG f uses x_1

$$k(X) = \underbrace{k(x_2, \dots, x_n)}_{\text{transc. basis.}}[x_1]/(f)$$

HEISUKE HIRONAKA

"Resolution of singularities of an algebraic variety over a field of characteristic 0."

Annals of Math.

Introduction

Let X be complex-(resp. real-)analytic space, i.e., an analytic **C**-(resp. **R**-)space in the sense defined in §1 of Chapter 0. *We ask if there exists a morphism of complex-(resp. real-)analytic spaces, say $f\colon \tilde{X} \to X$, such that:*

(1) *$\tilde{X}$ is a complex-(resp. real-)analytic manifold*, i.e., *a non-singular complex-(resp. real-)analytic space,*

(2) *if V is the open subspace of X which consists of the simple points of X, then $f^{-1}(V)$ is an open dense subspace of $\tilde{X}$ and f induces an isomorphism of complex-(resp. real-)analytic manifolds: $f^{-1}(V) \xrightarrow{\approx} V$, and*

(3) *f is proper*, i.e., *the preimage by f of any compact subset of X is compact in $\tilde{X}$.*

This is the problem which we call *the resolution of singularities in the category of complex-(resp. real-)analytic spaces*, or more specifically, the resolution of singularities of the given complex-(resp. real-)analytic space X. If X is a reduced *complex*-analytic space, then the open subspace V is dense in X and therefore the condition (2) implies that f is a *modification*. (The term *'reduced'* means that the structural sheaf of local rings has no nilpotent elements.) It should be noted, however, that V is not always dense if X is a reduced *real*-analytic space. So far as the resolution of singularities is concerned, we are particularly interested in the case of reduced complex-(resp. real-)analytic spaces. As for the general case in which X may not be reduced, we have a better formulation of the problem in terms of normal flatness. (See Definition 1, § 4, Ch. 0.)

The most significant result of this work is the solution of the above problem for the case in which X has *an algebraic structure*; that is to say, X is covered by a finite number of coordinate neighborhoods, each of