# Blow Ups

or: Zooming in

Two problems

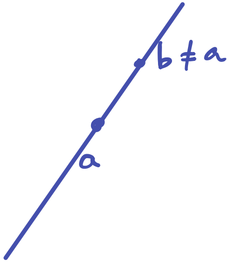
① Varieties have singularities

② Rational maps not def everywhere

$\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$

def. on $\mathbb{P}^n \setminus a$

No way to extend over $a$.

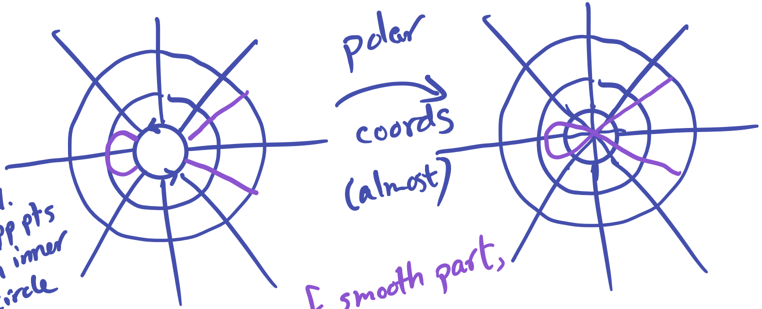



Blowup is a tool for fixing these.

Idea of blowup

Replace pt $p$ with set of lines thru $p$

Picture over $\mathbb{R}$:




Id. opp pts on inner circle

Möbius band

take preim of smooth part, then take closure singularity gone!

# The blowup of $\mathbb{A}^2$ at $O$

$$\pi : \mathbb{A}^n \setminus 0 \longrightarrow \mathbb{P}^{n-1}$$
$$(a_1, \dots, a_n) \longmapsto [a_1 : \dots : a_n]$$

$\Gamma_\pi \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ graph.

$\tilde{\mathbb{A}}^n = \overset{\text{Zar.}}{\text{closure}}$ of $\Gamma_\pi$ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

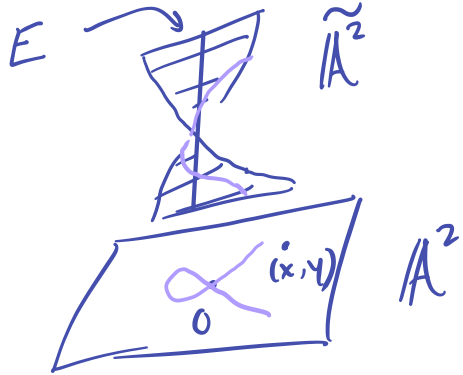
↳ blowup of $\mathbb{A}^n$ at $O$.

## $n=2$ case

$\pi(x,y) = [x:y]$ (or $x/y$)

$\tilde{\mathbb{A}}^2 = \{((x,y),[t_0:t_1]) : xt_1 = yt_0\}$

Check: this is the closure of $\Gamma_\pi$.




Projection to $\mathbb{A}^2$ induces

$$p : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$$

and $p^{-1}(x,y) = \begin{cases} ((x,y),[x:y]) & (x,y) \neq O \\ \underbrace{(0,0) \times \mathbb{P}^1}_{E \,=\, \text{exceptional line/divisor}} & (x,y) = O. \end{cases}$

**Fact.** $p$ induces $\tilde{\mathbb{A}}^2 \setminus E \xrightarrow{\sim} \mathbb{A}^2 \setminus 0$

# Affine cover of $\tilde{\mathbb{A}}^2$

$\mathbb{P}^1$ has std. aff. cover $U_0, U_1$

$\rightsquigarrow \tilde{\mathbb{A}}^2 = V_0 \cup V_1 \quad V_i \subseteq \mathbb{A}^2 \times \mathbb{A}^1$

where

$$V_0 = \{((x,y),[1:t_1]) : xt_1 = y\}$$
$$V_1 = \{((x,y),[t_0:1]) : x = yt_0\}$$

Note: $V_i \cong \mathbb{A}^2$

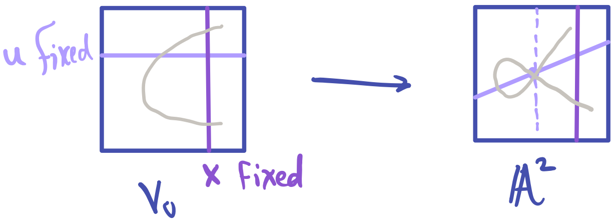
$V_0$ coords: $x, u = t_1$

$V_1$ coords: $y, v = t_0$

So $V_0 = \{((x, ux), [1:u])\} = \{(x,u)\}$

$V_1 = \{((vy, y), [v:1])\} = \{(y,v)\}$

Under $p : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$




Hor lines $\longrightarrow$ lines thru origin (get all but vertical)

Vert lines $\longrightarrow$ vert lines.

Similar for $V_1$.

## Resolving singularities

Say $X \subseteq \mathbb{A}^n$ sing. set $S$

A resolution is

$p: \tilde{X} \longrightarrow X$ s.t. $\tilde{X}$ nonsing

& restr. $\tilde{X} \setminus p^{-1}(S) \longrightarrow X \setminus S$

is an isomorphism.

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Resolution for

curves: blow up pts

surfaces over $\mathbb{C}$: Jung, Walker
Zariski '35

3-folds char $= 0$: Zariski
Annals '44

3-folds char $\neq 0$: Abhiyankar (Z's student)

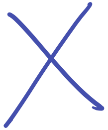
All varieties char 0: Hironaka ~'70

char $\neq 0$ open.

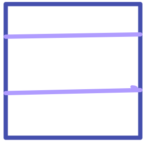
We'll look at curves ☺

## Example 1

$C = Z(x^2 - y^2)$



resolution:



in $\tilde{\mathbb{A}}^2$

Higher dim version:

$X = Z(x^2 + y^2 - z^2)$

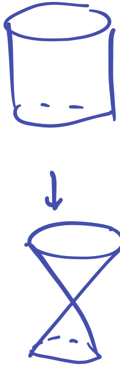
$\tilde{X} = Z(x^2 + y^2 - 1)$

$\tilde{X} \longrightarrow X$

$(x,y,z) \longmapsto (xz, yz, z)$

xy plane $\longmapsto$ pt

$\tilde{X}$



$X$

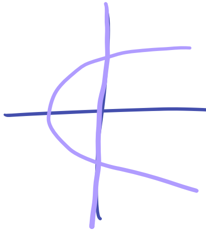
## Example 2

$C = Z(y^2 - x^2 - x^3)$

$p^{-1}(C) = \{((x,y),[t_0:t_1]) : y^2 = x^3 + x^2,\ t_0 y = t_1 x\}$

$p^{-1}(C) \cap V_0 = \{((x,xu),[1:u]) : x^2(x+1-u^2) = 0\}$

$= \{(x,u) : x^2(x+1-u^2) = 0\} \subseteq \mathbb{A}^2$



$E$

$p^{-1}(C) =$ parabola $\setminus$ pt

closure $\tilde{C}$ is parabola.

Smooth!

## Example 3

$C = Z(y^2 - x^3)$

$p^{-1}(C) \cap V_0 = \{(x,u) : \cancel{(xu)^2 = x^3}\}$

$x^2(x - u^2)$

$\rightsquigarrow$ parabola.

Aside: link of cusp is $(3,2)$-curve on $T^2$

(trefoil)

# Blowing up higher-dim subvars

## Algebra version:

Harris

$Y \subseteq X \subseteq \mathbb{A}^n$ aav's

$Y = Z(f_0, \dots, f_m)$ $f_i \in k[X]$

Define:

$\varphi : X \dashrightarrow \mathbb{P}^m$

$x \longmapsto [f_0(x) : \cdots : f_m(x)]$

regular on $X \setminus Y$

$\Gamma_\varphi \subseteq \mathbb{A}^n \times \mathbb{P}^m$ & $p : \Gamma_\varphi \to X$

closure is

$Bl_Y(X)$ blowup of $X$ at $Y$.

$p^{-1}(Y)$ "exceptional divisor"

example $0 = Y \subseteq X = \mathbb{A}^2$

$Y = Z(x,y)$

$\varphi : \mathbb{A}^2 \to \mathbb{P}^1$

$(x,y) \longmapsto [x:y]$

Can do similar for proj var's

(use homog. polys).

## Topological version:

Read in Harris.

Idea: replacing pts in $Y$ with space of normal directions.

e.g. $Y = Z$-axis in $\mathbb{A}^3$: pts in $Y$ get replaced with $\mathbb{P}^1$

**Thm** $X$ variety

$\varphi : X \dashrightarrow \mathbb{P}^n$ rat'l

Then $\exists$

$$\begin{tikzcd}
X_k \arrow[d] \arrow[dddr, "\text{morphism}"] & \\
\vdots \arrow[d, "\text{blowups}"'] & \\
X_1 \arrow[d] & \\
X = X_0 \arrow[r, dashed] & \mathbb{P}^n
\end{tikzcd}$$

So: a rat'l map is a reg map on some blowup.

Heisuke Hironaka

"Resolution of singularities of an algebraic variety over a field of characteristic 0."

Annals of Math.



## Introduction

Let $X$ be complex-(resp. real-)analytic space, i.e., an analytic **C**-(resp. **R**-)space in the sense defined in §1 of Chapter 0. *We ask if there exists a morphism of complex-(resp. real-)analytic spaces, say $f$: $\tilde{X} \to X$, such that*:

(1) *$\tilde{X}$ is a complex-(resp. real-)analytic manifold*, i.e., *a non-singular complex-(resp. real-)analytic space*,

(2) *if $V$ is the open subspace of $X$ which consists of the simple points of $X$, then $f^{-1}(V)$ is an open dense subspace of $\tilde{X}$ and $f$ induces an isomorphism of complex-(resp. real-)analytic manifolds*: $f^{-1}(V) \xrightarrow{\approx} V$, *and*

(3) *$f$ is proper*, i.e., *the preimage by $f$ of any compact subset of $X$ is compact in $\tilde{X}$.*

This is the problem which we call *the resolution of singularities in the category of complex-(resp. real-)analytic spaces*, or more specifically, the resolution of singularities of the given complex-(resp. real-)analytic space $X$. If $X$ is a reduced *complex*-analytic space, then the open subspace $V$ is dense in $X$ and therefore the condition (2) implies that $f$ is a *modification*. (The term '*reduced*' means that the structural sheaf of local rings has no nilpotent elements.) It should be noted, however, that $V$ is not always dense if $X$ is a reduced *real*-analytic space. So far as the resolution of singularities is concerned, we are particularly interested in the case of reduced complex-(resp. real-)analytic spaces. As for the general case in which $X$ may not be reduced, we have a better formulation of the problem in terms of normal flatness. (See Definition 1, § 4, Ch. 0.)

The most significant result of this work is the solution of the above problem for the case in which $X$ has *an algebraic structure*; that is to say, $X$ is covered by a finite number of coordinate neighborhoods, each of