Degree

Harris.

$X = Z(f)$ hypersurf.

$\rightsquigarrow$ $\deg X$ defined as

$\deg f$.

More generally:

$X \subseteq \mathbb{P}^n$ irred, k-dim

$\rightsquigarrow$ $\deg X$ is

① deg of any hypersurf in $\mathbb{P}^{k+1}$ birat eq to X

② the deg of a cover $X \to \mathbb{P}^k$

③ # pts of int. of generic $(n-k-1)$-plane with X

If X is a complex manifold

$\rightsquigarrow$ $[X] \in H_{2k}(\mathbb{P}^n; \mathbb{Z}) \cong \mathbb{Z}$

is the degree.

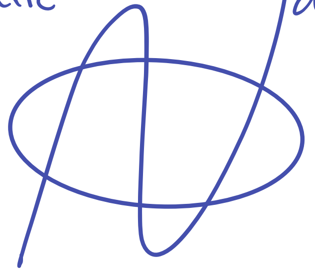
(also works for singular X).

Chapter 5 Curves or Bézout & applications.

k = alg closed.

B's thm. $C, D \subseteq \mathbb{P}^2$ curves of deg m, n, If C, D have no irred comp's in common then they intersect mn times with mult.

schematic



Special cases

① C, D lines $\rightsquigarrow$ 1 pt.

$\mathbb{P}^2$ exists so all lines intersect.

From this: all curves int. the right # of times.

(Like how solving $x^2+1=0$ allows to solve all polynomials)

② $C = Z(f)$ curve

$D = Z(ax+by+z)$ line.

Use $ax+by+z$ to elim. z from f

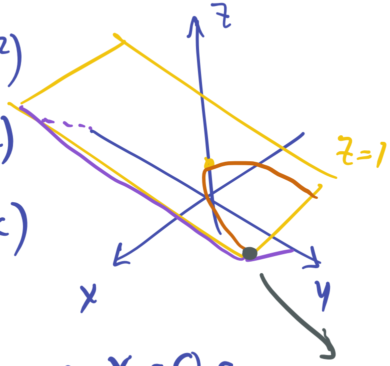
Passing to a generic affine chart, get a poly of $\deg = \deg f$ in one var.

Apply FTA.

you finish the proof of Bezout in this case.

example. $C = Z(yz - x^2)$

$D = Z(z - ax)$



$C \cap D = (yz - x^2, z - ax)$

$= (axy - x^2)$

Set $y=1$: $ax - x^2 = 0 \rightsquigarrow x = 0, a$

$\rightsquigarrow [0:1:0]$ & $[a:1:a^2]$

When $a=0$ get one pt of mult 2.

Special case of ②: Every conic meets line at ∞ in 2 pts w/ mult.

(cont.)

Special case of ②: Every conic meets line at ∞ in 2 pts w/ mult.

e.g. circle $(x-az)^2+(y-bz)^2=r^2z^2$

always contains $[1:i:0]$ & $[1:-i:0]$

If C, D both circles, they meet at those 2 pts plus 2 more in $\mathbb{A}^2$

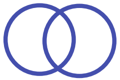
unless... concentric, in which case the 2 pts at ∞ have mult 2.

Similar: • hyperbola meets line at ∞ at the asymptotes

• parabola meets it at 1 pt with mult 2. (prev. ex. $a=0$)

example $C = Z(x^2+y^2-z^2)$

$D = Z((x-z)^2+y^2-z^2)$

circles 

you: find the 2 pts not at ∞.

Resultants

Goal: find common zeros of two polys

Say $f(x) = a_0 + \cdots + a_m x^m$

$g(x) = b_0 + \cdots + b_n x^n$

The resultant $\mathrm{Res}(f,g)$ is det of the $(m+n) \times (m+n)$ Sylvester matrix.

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_m & & \\ & a_0 & \cdots & \cdots & a_m & \\ & & \ddots & & & \ddots \\ & & & a_0 & \cdots & a_m \\ b_0 & \cdots & b_n & & & \\ & \ddots & & \ddots & & \\ & & b_0 & \cdots & b_n & \end{pmatrix} \begin{array}{l} \\ n \text{ times} \\ \\ \\ \\ m \text{ times} \\ \\ \end{array}$$

Prop. $\mathrm{Res}(f,g) = 0 \iff Z(f) \cap Z(g) \neq \emptyset$.

equiv: f,g no common factors.

Linear case

$$a_0 + a_1 x = 0$$
$$b_0 + b_1 x = 0.$$

$$\rightsquigarrow \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \quad \checkmark$$

Quadratic case

$$a_0 + a_1 x + a_2 x^2 = 0$$
$$b_0 + b_1 x + b_2 x^2 = 0$$

$$\rightsquigarrow \begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix}$$

Note

$(1, x, x^2)$ solves 1^{st} eqn

$\Leftrightarrow (x, x^2, x^3)$ solves second.

Can see rank ≥ 3 (look at 1^{st} 3 rows)

$\Rightarrow \dim \ker \leq 1$

So $\det = 0 \Rightarrow \dim \ker = 1$

Observe: Can artificially make 2 new eqns

$$a_0 x + a_1 x^2 + a_2 x^3$$
$$b_0 x + b_1 x^2 + b_2 x^3$$

Now we have 4 lin eqns in the "variables" x, x^2, x^3

Take a vector in null space of Sylv. with first entry 1

$$\begin{pmatrix} a_0 & a_1 & a_2 & 0 \\ 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = 0$$

Prop. $\mathrm{Res}(f,g)=0 \iff f,g$ have common root.

Pf. Say α = root of f & g.

$\rightsquigarrow \exists$ polys f_1, g_1 of deg $m-1, n-1$

s.t. $f(x) = (x-\alpha) f_1(x)$

$g(x) = (x-\alpha) g_1(x)$

$$\Rightarrow f(x) g_1(x) - g(x) f_1(x) = 0.$$

Both terms have deg $m+n-1$.

Equating coeffs to 0 gives

$m+n$ lin eqns in $m+n$ vars.

(coeffs of f_1, g_1)

The matrix is the Sylv. matrix.

The existence of a soln shows $\mathrm{Res}(f,g)=0$.

Conversely, say $\mathrm{Res}(f,g)=0$. As above get soln f_1, g_1 to

$$f(x) g_1(x) - g(x) f_1(x) = 0.$$

A root α of f must be a root of g or f_1. If g, done.

If f_1, cancel $(x-\alpha)$ from f & f_1 continue inductively.

$\square$

In proj space

$C = Z(f)$

$D = Z(g)$

Assume WLOG $[0:0:1]$ on neither.

($\exists$ purely z term)

$\rightsquigarrow f(x,y,z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$

$g(x,y,z) = z^n + \cdots + b_0$

a_i, b_i homog polys in x,y of deg $m-i$, $n-i$

$\rightsquigarrow R(x,y)$ resultant wrt z

poly in x,y.

Prop. $R(x,y)$ either $\equiv 0$ or deg mn.

Example. $f(x,y,z) = x^2+y^2-z^2$

$g(x,y,z) = x^3 - x^2z - xz^2$

$\rightsquigarrow R(x,y) = -x^2y^4 \rightsquigarrow x=0$ or $y=0$

2 roots w/ mult

4 roots w/mult.

$x=0: y^2-z^2=0 \rightsquigarrow [0:1:1], [0:1:-1]$

both have mult 2. $\longrightarrow$ $y=0: x^2-z^2=0 \rightsquigarrow [1:0:1], [1:0:-1]$