Resultants

$f(x) = a_0 + a_1 x + \cdots + a_m x^m$

$g(x) = b_0 + \cdots + b_n x^n$

$\rightsquigarrow$ Sylvester matrix

$$\begin{pmatrix} a_0 & \cdots & a_m & & \\ & \ddots & & \ddots & \\ & & a_0 & \cdots & a_m \\ b_0 & \cdots & b_n & & \\ & \ddots & & \ddots & \\ & & b_0 & \cdots & b_n \end{pmatrix} \begin{matrix} \text{n times} \\ \\ \\ \text{m times} \\ \\ \\ \end{matrix}$$

det. is $\mathrm{Res}(f,g)$.

Prop. $\mathrm{Res}(f,g) = 0 \iff$ common factor.

Lemma. f, g have common factor $\iff$

$\exists\, s, t$: $\deg s < \deg g$

$\deg t < \deg f$

$fs + gt = 0$.

Pf. ($\Rightarrow$) f, g have common factor

$\Rightarrow f = hf_1 \quad g = hg_1$

$\Rightarrow fg_1 - gf_1 = 0$.

($\Leftarrow$) $fs + gt = 0$. Assume no comm. fac.

$\Rightarrow$ roots of f are roots of gt

$\Rightarrow$ roots of f are roots of t

but $\deg t < \deg f$. $\square$

Prop. $\operatorname{Res}(f,g)=0 \iff$ common factor.

Lemma. f, g have common factor $\iff$

$\exists\, s,t:$ $\deg s < \deg g$

$\deg t < \deg f$

$fs + gt = 0.$

Pf of Prop. Want to know existence of s,t as in Lemma.

Let $P(x) = f(x)\, s(x) + g(x)\, t(x)$

$$s = \sum_{i<n} s_i x^i \quad t = \sum_{i<m} t_i x^i$$

$$\rightsquigarrow P(x) = (a_m s_{n-1} + b_n t_{m-1}) x^{m+n-1} + \cdots$$

Solving for $(s_0, \ldots, s_{n-1}, t_0, \ldots, t_{m-1})$

or:

$$(s_0, \ldots, s_{n-1}, t_0, \ldots, t_{m-1}) \cdot \begin{pmatrix} a_0 & \cdots & a_m & & \\ & \ddots & & \ddots & \\ & & a_0 & \cdots & a_m \\ b_0 & \cdots & b_n & & \\ & \ddots & & \ddots & \\ & & b_0 & \cdots & b_n \end{pmatrix} = 0$$

$\square$

Curves in $\mathbb{P}^2$ version

$C = Z(f)$, $D = Z(g)$

WLOG $[0:0:1] \notin C \cup D$.

$\Rightarrow$ f, g have Z-only term, which must be lead term.

$\rightsquigarrow$ $f = a_0 Z^m + a_1 Z^{m-1} + \dots$

$g = b_0 Z^n + b_1 Z^{n-1} + \dots$

} deg m, n as poly's in Z, coeffs in $k[x,y]$.

a_i, b_i homog. deg i in x, y.

$\rightsquigarrow R(x,y) = \mathrm{Res}(f,g)$ poly in x, y.

Prop. $R(x,y)$ is either: $\equiv 0$ or homog. of deg mn.

Pf. To show $R(tx,ty) = t^{mn} R(x,y)$

$$R(tx,ty) = \begin{pmatrix} a_0 & ta_1 & t^2a_2 & \cdots & \\ & ta_0 & t^2a_1 & \cdots & \\ & & & & \\ b_0 & tb_1 & \cdots & & \\ & tb_0 & t^2b_1 & \ddots & \end{pmatrix}$$

Mult. rows by $t^0, t^1, \dots, t^{n-1}, t^0, \dots, t^{m-1}$

factor: $n(n-1)/2 + m(m-1)/2$

Divide cols by $t^0, \dots, t^{m+n-1}$ $\rightsquigarrow \mathrm{Res}(f,g)$

factor: $(m+n)(m+n-1)/2$

Difference is mn. $\square$

Bézout's Thm $C, D \subseteq \mathbb{P}^2$ curves of deg m, n w/ no common irred comp. Then they intersect mn times with mult.

Pf. **Setup**

① Suffices to consider C, D irred.

② $\dim C \cap D = 0$

$\Rightarrow |C \cap D| < \infty$.

③ WLOG change coords so $x \neq 0$ at all pts of $C \cap D$.

④ Say $C = Z(f)$, $D = Z(g)$

$\rightsquigarrow R(x, y)$

Step 1. $R(x,y)$ homog. of deg mn

If $R(x,y) \equiv 0$ then $\forall [a:b] \in \mathbb{P}^1$ f, g have common 0, violating ②.

Apply Prop.

Write $R(x,y) = x^{mn} R_*(y/x)$ where R_* is poly in $t = y/x$ of $\deg \leq mn$.

Step 2. $\deg R_* = mn$.

$\deg R_* < mn \Leftrightarrow$ no y^{mn} term. $\Leftrightarrow$ all terms of R have x $\Leftrightarrow R(0,1) = 0$ violates ③.

Step 3 {Roots of R_*}

$\longleftrightarrow$ $C \cap D$ w/ mult.

($\rightarrow$) If α is a root of R_*

then $\alpha = a/b$ with $R(a,b) = 0$.

$\Rightarrow$ $f(a,b,z)$, $g(a,b,z)$ have

common root c $\rightsquigarrow$ pt of $C \cap D$ of form $[a:b:c]$.

($\leftarrow$) $[a:b:c] \in C \cap D$ $a \neq 0$

$\Rightarrow$ b/a root of R_*

Need Setup ⑤ No two pts $[a:b:c]$, $[a:b:c'] \in C \cap D$. $\Longleftrightarrow$ no pts of $C \cap D$ lie on line $\parallel$ to z-axis.

No 2 pts of $C \cap D$ colin with $[0:0:1]$

$\square$ $\Updownarrow$

Define multiplicity now :)

~~# common roots c w/mult. corresp. to given α.~~

~~or deg of c as root of f-g @ (a,b).~~

or more prec.

~~Claim. This equals deg. of α as root of R_*~~

Mult. defined as mult of root of R_*.

Right defn of mult

Fitchett.

Let $p \in C \cap D$

Assume p in std aff. chart $Z = 1$.

Define

$$i(C \cap D, p) = \dim_k \left(\frac{\mathcal{O}_p}{(f,g)_p} \right)$$

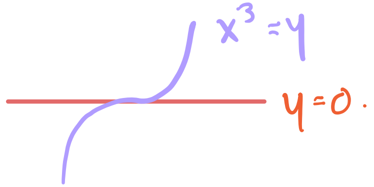
$\mathcal{O}_p$ = rat'l fns def. at p.

$(f,g)_p$ = ideal gen by f, g in $\mathcal{O}_p$.

"localization": allow denominators that don't vanish at p

i.e. denoms don't lie max ideal at p which is $(x_1 - p_1, \dots, x_n - p_n)$.

Example 1. $k = \mathbb{C}$



$f(x,y) = y - x^3$

$g(x,y) = y$

exercise.

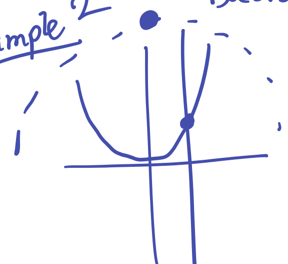
$$\frac{\mathcal{O}_p}{(f,g)_p} = \frac{\mathbb{C}[x,y]_{(x,y)}}{(y - x^3, y)_{(x,y)}} \cong \left(\mathbb{C}[x,y] / (y - x^3, y) \right)_{(x,y)}$$

$$\cong \left(\mathbb{C}[x] / (x^3) \right)_{(x)} \cong \mathbb{C}^3 \text{ as } \mathbb{C}\text{-v.s.}$$

basis $1, x, x^2$

exercise.

Example 2



Why are the multiplicities the same?

Write $I_p(C,D)$.
or $I_p(f,g)$

Fulton

Gims (?)

Axioms

① $I_p(f,g) = I_p(g,f)$

② $I_p(f,g) = \begin{cases} \infty & p \text{ in a common comp.} \\ 0 & p \notin C \cap D \\ \in \mathbb{N} & \text{otherwise.} \end{cases}$

③ C, D lines, $p \in C \cap D \Rightarrow I_p(f,g) = 1$

④ $I_p(f_1 f_2, g) = I_p(f_1, g) + I_p(f_2, g)$

⑤ $I_p(f,g) = I_p(f, g+fh)$ if $\deg h = \deg g - \deg f$.

Thm. An $I_p(f,g)$ satisfying the axioms exists and is unique.