Final HW

7 problems on web site

or project on Teams 1-2 "pages"

Learn something & tell us about it.

Ideas: Robotics

Splines

Image deblurring.

Gröbner bases

Divisors.

Riemann-Roch Thm

Fano varieties

Poncelet's porism

Hironaka's thm

Sheaves / Schemes.

or creative writing
artwork.

Computing multiplicities

$C = Z(x^2+y^2-z^2)$

$D = Z(x^2+y^2-2z^2)$

$C \cap D = [\pm i : 1 : 0]$

Via axioms

$I_p(x^2+y^2-z^2,\ x^2+y^2-2z^2)$

$= I_p(x^2+y^2-z^2,\ z^2)$ "row op"

$= I_p(x^2+y^2,\ z^2)$

$= 2I_p(x^2+y^2,\ z)$

$= 2I_p(x+iy, z) + 2I_p(x-iy, z)$

$= 2+0$ or $0+2$ "lines"

dep. on p

Via resultant

$$R(x,y) = \det\begin{pmatrix} -1 & 0 & x^2+y^2 & 0 \\ 0 & -1 & 0 & x^2+y^2 \\ -2 & 0 & x^2+y^2 & 0 \\ 0 & -2 & 0 & x^2+y^2 \end{pmatrix}$$

$= (x^2+y^2)^2$

$\rightsquigarrow R_*(t) = (1+t^2)^2$

$= (1+it)^2(1-it)^2$

via Local rings

$f = Z(y - x^2) \quad g = Z(y)$

$$\mathbb{C}[x,y]_{(x,y)} \Big/ (f,g)_{(x,y)}$$

$$\cong \left(\mathbb{C}[x,y] \big/ (f,g) \right)_{(x,y)}$$

$$\overset{\text{as a set}}{=} \left\{ \frac{ax+b}{cx+d} : d \neq 0 \right\}$$

WTS : dim = 2

basis $1, x$.

rationalize:

$$\frac{ax+b}{cx+d} \cdot \frac{-cx+d}{-cx+d}$$

$$= \frac{-acx^2 + (ad-bc)x + bd}{d^2 - c^2x^2}$$

$$= \left(\frac{ad-bc}{d^2} \right) x + \frac{b}{d}$$

exercise. Do example on last slide this way.

easier. Fitchett example $x^3 = y$, $y = 0$.

Easy conseq's of Bézout

① If $|C \cap D| = mn$, all intersections are transverse (mult 1)

② If $|C \cap D| > mn$, common irred. comp.

③ Any two proj. curves intersect.

④ $|C \cap L| = m$ with mult. (L line).

More conseq's

⑤ C irred has at most $\binom{d-1}{2}$ sing pts.

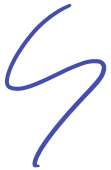
Gathmann Prop 13.5 $\qquad d = \deg C$

Fulger Cor 8.14

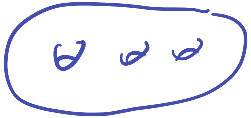
⑥ Degree genus formula.

C smooth $\Rightarrow g = \binom{d-1}{2}$

Kerr Sec 14.3.

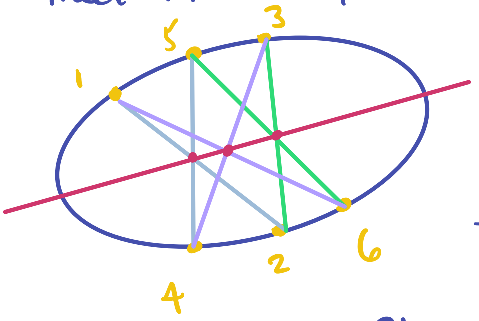


really

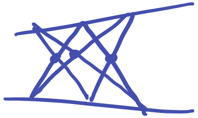


⑦ Pascal's mystic hexagon.

If a hexagon is inscribed in an irred conic then opp sides meet in colin pts.



Pappus



Stevens Prop 3.2.1

⑧ Cayley - Bacharach Thm

Kerr Thm 15.12
Cor 15.2.2

C, D cubic $|C \cap D| = 9$

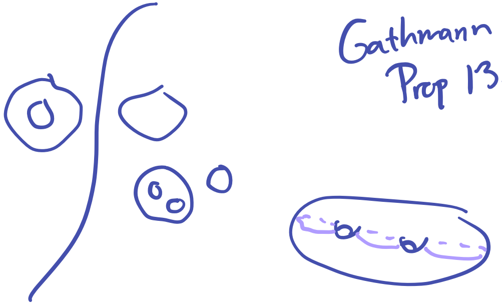
Any cubic passing thru 8 of the pts, passes thru the 9th

⑨ Space of smooth? curves of deg d in $\mathbb{P}^2$ is a complex proj space of dim $d(d+3)/2$

Moonen p.240

⑩ Harnack's thm

A smooth real proj. curve in $\mathbb{P}^2$ has at most $\binom{d-1}{2}+1$ loops



Gathmann Prop 13.11

(11) A nonsing cubic has 9 pts of inflection

Gims Cor 5.4

(12) $\mathrm{Aut}\,\mathbb{P}^n \cong GL_{n+1}\,k$. Gathmann Prop 13.4

Classific. of conics: 

(13) Classific. of irred singular cubics.

$y^2 - x^3 - x^2$ 

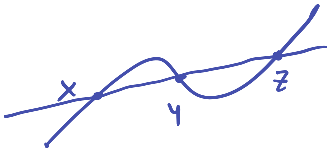
$y^2 - x^3$ 

and smooth cubics:

$y^2 = 4x^3 - ax - b$  or 

Hulek Prop 4.11

(14) Smooth cubics are groups!



Dolgachev p.52

Mystic Hexagon

Prop. Say C, D deg n

$|C \cap D| = n^2$

Assume mn of the int pts lie on irred curve E of deg m

Then the remaining $n(n-m)$ pts lie on a curve F

of $\leq n-m$.

Other pf Shafarevich

Pf of Mystic Hex

Say vertices are $p_0, \dots, p_5$

Let L_i = line thru $p_i p_{i+1}$ (mod 5)

Let $L = L_0 L_2 L_4$ $L' = L_1 L_3 L_5$ cubics.

L & L' have no common factor.

Bézout $\Rightarrow |L \cap L'| \leq 9$.

If < 9, nothing to do.

6 pts of $L \cap L'$ are $p_0, \dots, p_5$

which lie on conic.

The other 3 line on a line by the Prop. $\square$

Prop. Say C, D deg n

$|C \cap D| = n^2$

Assume mn of the int pts lie on irred curve E of deg m

Then the remaining $n(n-m)$ pts lie on a curve F of deg $\leq n-m$.

Pf. Say C, D, E given by f, g, h.

Let $[a:b:c] \in E \setminus (C \cap D)$

Let $F_0 = Z(p)$

$$p = g(a,b,c) f - f(a,b,c) g$$

deg $\leq n$.

Then $|E \cap F_0| \geq mn + 1$ b/c it contains mn pts of $E \cap (C \cap D)$ and $[a:b:c]$.

Bézout $\Rightarrow F_0, E$ have common comp.

E irred, so it is a comp. of F_0

$$\Rightarrow p = \overset{E}{h} q \qquad \deg q \leq n-m$$

Let $F = Z(q)$

Each $[u:v:w]$ in $(C \cap D) \setminus E$ satisfies

$f = 0, g = 0$ thus satisfies $p = 0$

Also $h(u,v,w) \neq 0 \Rightarrow q(u,v,w) = 0$.

i.e. $[u:v:w] \in F$. $\square$