Curves thru given pts

Terry Tao (Husmöller)

Existence

① Thru 2 pts $\exists$ line.

$ax + by = c$

2 constraints (given pts)
3 unknowns a, b, c.

② Thru 5 pts $\exists$ quadric

(same lin. alg)

Bézout $\Rightarrow$ if 3 are collinear

then quadric is reducible (not a conic) $\rightsquigarrow$ union of 2 lines.

③ Thru 9 pts $\exists$ cubic (same lin alg).

Uniqueness

Need some kind of general posn.

① always unique (if 2 pts distinct)

② if all 5 pts collinear, then can take that line & any other line.

BUT if no 3 collinear get uniqueness: can't be 2 distinct lines by hyp.
can't be two different quadrics by Bezout.

(3) If 8 of 9 pts lie on conic C then many $C \cup L$ are cubics containing the 9 pts.

Even without this, uniqueness harder to come by.

Say $C_0 = Z(f_0)$, $C_\infty = Z(f_\infty)$ cubics.

& $|C_0 \cap C_\infty| = 9$.

Then $C_t = Z(f_0 + t f_\infty)$ $t \in k \cup \infty$ contains all 9 pts.

But these are only ones going through the 9 pts, or even 8 of them.

<u>Cayley-Bacharach thm</u> k alg closed.

If D is a cubic curve passing thru 8 pts of $C_0 \cap C_\infty$ then $D = C_t$ some t. In partic, D passes thru the $9^{\underline{th}}$ pt.

Pf. Assume $\exists$ D passing thru 8 pts $a_1, \dots, a_8$ of $C_0 \cap C_\infty$.

Say $C_0 = Z(p_0)$ $C_\infty = Z(p_\infty)$

$D = Z(p)$ Want: $D = C_t$

Assume $D \neq C_t$ any t. (*)

Claim 1. No 4 of the a_i collin.

Pf. Bézout $\Rightarrow$ C_0, C_∞ would both contain this line.

$\Rightarrow |C_0 \cap C_\infty| = \infty > 9$.

Claim 2. No 7 of the a_i lie on quadric.

Pf. Same.

Claim 3. Any 5 of the a_i determine a unique quadric

Pf. If 5 pts lie on two quadrics E, F

Bezout $\Rightarrow$ $E \cap F$ contains line L

Claim 1 $\Rightarrow$ L contains at most 3 of the a_i.

The other ≥ 2 pts must lie on other comp of E (line) & other comp of F. (line)

Both are lines. Must be same line. So $E = F$.

Claim 4. No 3 of the a_i collin.

Pf. Say $a_1, a_2, a_3 \in L$ line

Claim 1 $\Rightarrow$ $a_i \notin L$ $i > 3$.

$a_4, \ldots, a_8$ lie on unique quadric E.

(Claim 3)

Let b be another pt on L

c be another pt not on E or L.

By lin alg $\exists$ cubic

$$q = x p + y p_0 + z p_\infty$$

vanishing at b, c. (3 vars, 2 eqns)

Fact A linear interp b/w C_0, C_∞, D cont. $a_1, \ldots, a_8$

By ⊛ $q \neq 0$.

Let $F = Z(q)$.

$F \cap L$ contains a_1, a_2, a_3, b

Bezout $\Rightarrow$ $F = L \cup$ quadric

The quadric contains

$a_4, \ldots, a_8$ (p, p_0, p_∞ all vanish at $a_1, \ldots, a_8$)

By uniqueness of E:

$$F = L \cup E$$

but c not in E, L hence not in F. contradiction

$\square$

Claim 5. No 6 of $a_1, \dots, a_8$ lie on a quadric.

Pf. Say $a_1, \dots, a_6$ lie on Q = quadric.

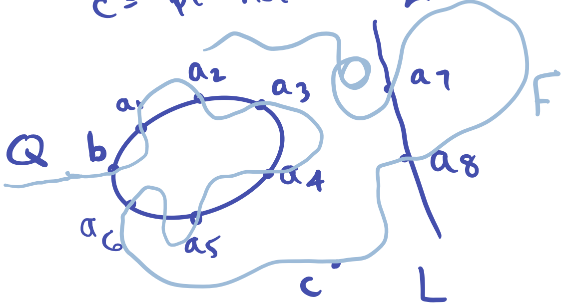
Claim 4 $\Rightarrow Q \neq L_1 \cup L_2$

$\Rightarrow Q$ conic.

Let L = line thru a_7, a_8.

b = another pt on Q

c = pt not on L or Q



As before, have nonzero cubic

$$q = xp + yp_0 + zp_\infty$$

vanishes at b, c. Also at $a_1, \dots, a_8$

Let $F = Z(q)$ Note $b, c \in F$.

F contains $a_1, \dots, a_6, b \in Q$

$\Rightarrow F = Q \cup$ line

The line is L, but L hence F does not contain c.

Contrad.

Finishing...

Let L = line thru a_1, a_2

Q = quadric thru $a_3, \ldots, a_7$

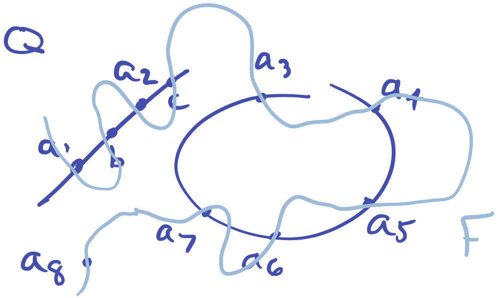
Claim 3 $\Rightarrow$ Q unique

Claim 4 $\Rightarrow$ Q conic (can't be 2 lines)

Claim 4 $\Rightarrow$ $a_8 \notin L$

Claim 5 $\Rightarrow$ $a_8 \notin Q$

Let $b, c \in L \setminus Q$



Again $\exists$ non-0 cubic

$q = xp + yp_0 + zp_\infty$

vanishing on b, c $\rightsquigarrow$ $F = Z(q)$

$F \cap L$ contains a_1, a_2, b, c

Bézout $\Rightarrow$ $F = L \cup$ quadric

The quadric contains $a_3, \ldots, a_7$

so it is Q

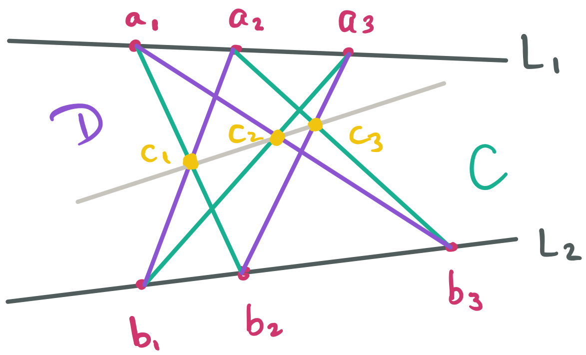
So $F = L \cup Q$

$\Rightarrow$ a_8 not in F.

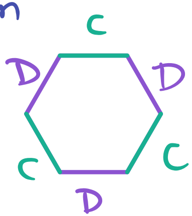
But F is a lin interp. of 3 cubics cont. $a_1, \ldots, a_8$.

contradiction. $\square$

Proof of Pappus



C, D are cubics given by triples of lines in hexagon.



E given by L_1, L_2, line thru c_1, c_2

Cayley-Bacharach $\Rightarrow$ E contains c_3

(We assumed $c_1 \neq c_2$, o/w nothing to prove.)

Pascal's Mystic Hexagon similar

(note: the c_i can't all lie on the conic by Bezout.)

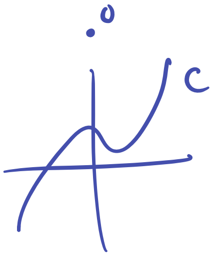
Smooth cubics are groups

$C = Z(y^2 - x^3 - ax - b) \subseteq \mathbb{P}^2$

smooth

$O = [0:1:0] \in C$

pt at ∞



For $c = [u:v:w]$

let $\bar{c} = [u:-v:w]$ refl. thru x-axis. in $\mathbb{A}^2$ plane

so $\bar{O} = O$.

Define $a + b = \bar{c}$

__Thm.__ C is an abel. gp.

__Pf.__ identity: O.

inverse: $c + \bar{c} = O$.

abelian: ✓

associativity: assume WLOG

$o, a, b, c, a+b, b+c, -(a+b),$

$-(b+c)$ all distinct from

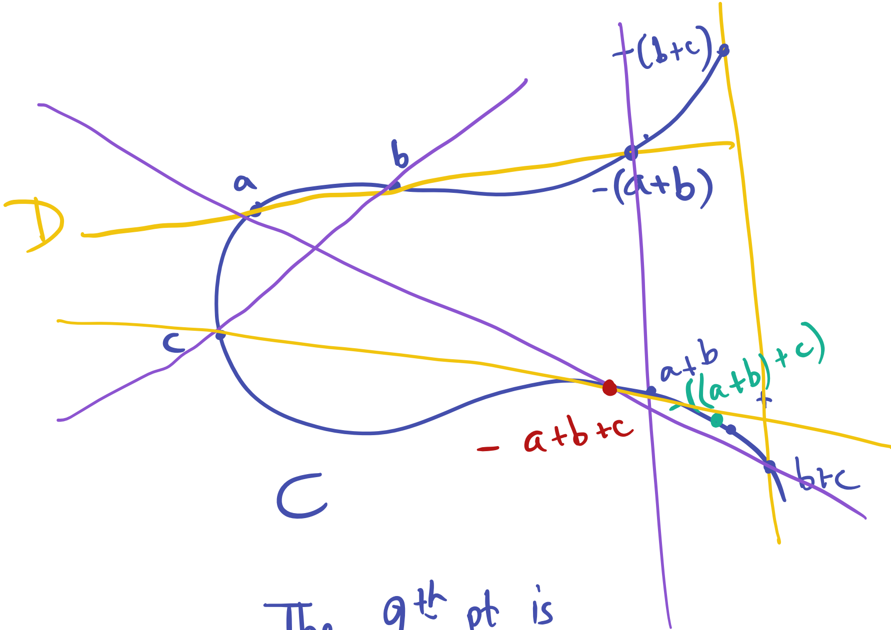
each other and

$-((a+b)+c)$ & $-(a+(b+c))$

(uses smoothness)

Let $D = \overline{ab}, \overline{c(a+b)}, \overline{o(b+c)}$

$E = \overline{o\,a+b}\ \ \overline{bc}\ \ \overline{a(b+c)}$



C & D cubics meeting at 9 pts,

no common comp.

E passes thru 8 hence 9^{th} by Cay-Ba

The 9^{th} pt is

$-((a+b)+c)$

The line thru $a, b+c$ meets C

in $-(a+(b+c))$ & $-((a+b)+c)$

hence equal. $\square$

Tao says: Pascal is a degen. case of the assoc. law on cubic

& Pappus is a degen. case of Pascal

Mordell's Thm. $\mathbb{Q}$ pts on C form a fin. gen. abel. gp.