Curves thru given pts

Existence

1. Thru 2 pts 1 line.
   \[ ax + by = c \]
   2 constraints (given pts)
   3 unknowns \(a, b, c\).

2. Thru 5 pts 1 quadric
   (same lin. alg)
   Bézout \(\Rightarrow\) if 3 are collinear
   then quadric is reducible
   (not a conic) \(\Rightarrow\) union of 2 lines.

Uniqueness

Need some kind of general posn.

1. always unique (if 2 pts distinct)
2. if all 5 pts collinear, then can take that line & any other line.

BUT if no 3 collinear get uniqueness: can't be 2 distinct lines by hyp.

3. Thru 9 pts 1 cubic (same lin alg).

\(\text{Bezout} \neq \text{if 3 are collinear} \Rightarrow \text{quadric is reducible}
\text{not a conic} \Rightarrow \text{union of 2 lines.}\)
3. If there are 8 of 9 pts lie on conic C, then many CUL are cubics containing the 9 pts.

Even without this, uniqueness harder to come by.

Say $C_0 = \mathbb{Z}(f_0)$ cubics.

$C_\infty = \mathbb{Z}(f_\infty)$

& $|C_0 \cap C_\infty| = 9$.

Then $C_t = \mathbb{Z}(f_0 + tf_\infty) \text{ t} \in kv\infty$ contains all 9 pts.

But these are only ones going through the 9 pts, or even 8 of them.

Cayley - Bacharach thru k alg closed.

If $D$ is a cubic curve passing thru 8 pts of $C_0 \cap C_\infty$ then $D = C_t$ some t. In partic, $D$ passes thru the 9th pt.
Claim 1. 

No 4 of the ai collin.

Proof. Bézout \( \Rightarrow \) Co, Coo would both contain this line.

\[ \Rightarrow |C_0 \cap C_\infty| = \infty > 9. \]

Claim 2. 

No 7 of the ai lie on quadric.

Proof. Same.

Claim 3. 

Any 5 of the ai determine a unique quadric

Proof. If 5 pts lie on two quadrics E, F

Bézout \( \Rightarrow \) EnF contains line L

Claim 1 \( \Rightarrow \) L contains at most 3 of the ai.

The other \( \geq 2 \) pts must lie on other comp of E (line)
& other comp of F. (line)

Both are lines. Must be same line. So E = F.
Claim 4. No 3 of the $a_i$ collin.

Proof. Say $a_1, a_2, a_3 \in L$ line

Claim 1 $\Rightarrow a_i \not\in L $ $i > 3$.

$a_4, ..., a_8$ lie on unique quadric $E$.

(Claim 3)

Let $b$ be another pt on $L$.

$c$ be another pt not on $E$ or $L$.

By lin alg $E$ cubic

$q = x p + y p_0 + z p_0 o$

Vanishing at $b, c$. ($3$ vars, $2$ eqns)

By $\otimes q \neq 0$.

Let $F = \mathbb{Z}(q)$.

Fn $L$ contains $a_1, a_2, a_3, b$

Bezout $\Rightarrow F = L \cup$ quadric

The quadric contains $a_4, ..., a_8$ ($p, p_0, p_{oo}$ all vanish at $a_1, ..., a_8$)

By uniqueness of $E$:

$F = L \cup E$

but $c$ not in $E, L$ hence not in $F$. contradiction.
Claim 5. No 6 of $a_1, \ldots, a_8$ lie on a quadric.

*Proof.* Say $a_1, \ldots, a_6$ lie on $Q = \text{quadric}$.

Claim 4 $\implies$ $Q \not\subset L_1 \cup L_2$

$\implies$ $Q$ conic.

Let $L$: line thru $a_7, a_8$.

$b =$ another pt on $Q$

c = pt not on $L$ or $Q$

As before, have nonzero cubic

$$q = xp + yp + zp_0$$

vanishes at $b, c$. Also at $a_1, \ldots, a_8$

Let $F = \mathbb{Z}(q)$. Note $b, c \in F$.

$F$ contains $a_1, \ldots, a_6, b \in Q$

$\implies F = Q \cup \text{line}$

The line is $L$, but $L$ hence $F$ does not contain $c$.

Contrad.
Finishing...

Let \( L = \text{line thru } a_1, a_2 \)
\( Q = \text{quadric thru } a_3, \ldots, a_7 \)

Claim 3 \( \Rightarrow Q \text{ unique} \)
Claim 4 \( \Rightarrow Q \text{ conic (can't be 2 lines)} \)
Claim 4 \( \Rightarrow a_8 \notin L \)
Claim 5 \( \Rightarrow a_8 \notin Q \)

Let \( b, c \in L \setminus Q \)

Again, \( J \text{ non-0 cubic} \)
\( q = xP + yP_0 + zP_00 \)
Vanishing on \( b, c \mapsto F = Z(q) \)

\( F \cap L \text{ contains } a_1, a_2, b, c \)
Bézout \( \Rightarrow F = L \cup Q \quad \text{quadric} \)
The quadric contains \( a_3, \ldots, a_7 \)

So it is \( Q \)
So \( F = L \cup Q \)

\( \Rightarrow a_8 \text{ not in } F. \)
But \( F \) is a lin interp. of 3 cubics cont. \( a_1, \ldots, a_8 \)
contradiction.
Proof of Pappus

Cayley-Bacharach \implies E contains $C_3$

(We assumed $C_1 \neq C_2$, o/w nothing to prove.)

Pascal's Mystic Hexagon similar

(note: the $C_i$ can't all lie on the conic by Bezout.)

$C, D$ are cubics given by triples of lines in hexagon.

$E$ given by $L_1, L_2$, line thru $c_1, c_2$
Smooth cubies are groups

\[ C = \mathbb{Z}(y^2 - x^3 - ax - b) \leq \mathbb{P}^2 \]

smooth

\[ o = [0:1:0] \in C \]

pt at \( \infty \)

For \( c = [u:v:w] \)

let \( \bar{c} = [u:-v:w] \)

refl. thru x-axis.
in \( \mathbb{A}^2 \) plane

so \( \bar{0} = 0 \).

Define \( a + b = \bar{c} \)

Thm. \( C \) is an abel. gp.

PF: identity: \( 0 \).

inverse: \( c + \bar{c} = 0 \).

abelian: \( \checkmark \).
associativity, assume WLOG
\(0, a, b, c, a+b, b+c, -(a+b), -(b+c)\) all distinct from each other and \
\(-(a+b)+c) & -(a+(b+c))

(uses smoothness)

Let \(D = ab, c(a+b), o(b+c)\)
\(E = oab, bc, a(b+c)\)

\(C & D\) cubics meeting at 9 pts, no common comp. \(E\) passes thru 8 hence 9th by Cay-Ba

The 9th pt is \(-(a+b)+c)\)
The line thru \(a, b+c\) meets \(C\) in \(-(a+(b+c)) & -(a+b+c)\) hence equal. \(\Box\)
Tao says: Pascal is a degenerate case of the associativity law on cubic.

& Pappus is a degenerate case of Pascal.

Mordell’s Thm. \( \mathbb{Q} \) pts on \( C \)

form a fin. gen. abel. gp.