Goal: Classify cubic curves.

Hulek

i.e. $C = Z(f) \subseteq \mathbb{P}^2$

deg 3 (char $k = 0$)

proj. equiv: $GL_3 k$

4 cases:
1. 3 lines
2. conic + line
3. sing irred
4. smooth irred cubic.

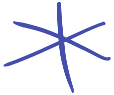
Case 1 3 lines.

lines in $\mathbb{P}^2 \longleftrightarrow$ pts in $\mathbb{P}^2$

via orthog. compl. in k^3

Prop. $C =$ union of 3 lines

Then C is proj eq to exactly one of

1. $Z(xyz)$ 
2. $Z(xy(x+y))$ 
3. 
4. 

Pf. Translate to problem about pts in $\mathbb{P}^2$

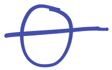
1. $\Leftrightarrow$ collinear, distinct...

$\square$

Case 2: Conic + line

Prop. C = conic + line = $Q \cup L$

The C is proj eq to exactly one of

① $Z((xz-y^2)y)$ 

② $Z((xz-y^2)x)$ 

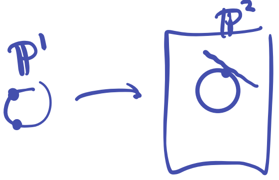
Pf. We already showed (using quad. forms) Q is proj equiv to $Z(x^2+y^2+z^2)$

$$\sim Z(xz-y^2)$$

Bézout $\Rightarrow$ 2 cases

① $|Q \cap L| = 2$

② $|Q \cap L| = 1$



Q is image of $\mathbb{P}^1 \to \mathbb{P}^2$

Up to linear change of coords in $\mathbb{P}^1$ can assume int. pts are

① $[1:0:0]$ & $[0:0:1]$

② $[0:0:1]$

Show $\exists$ linear change of coords on $\mathbb{P}^2$ realizing this reparameterization.

L is hence determined. $\square$

If the param of Q is

$$[t:u] \longmapsto [t^2 : tu : u^2]$$

If the re-parametrization in $\mathbb{P}^1$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

the reparam. in $\mathbb{P}^2$ is

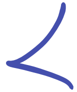
$$\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

Case 3 Sing. irred. cubics.

Prop. C = sing. irred cubic.

Then C is proj equiv to exactly one of

① $Z(y^2z - x^3 - x^2z)$ 

② $Z(y^2z - x^3)$ 

Fact. $X = Z(f) \subseteq \mathbb{P}^2$

$p \in X \quad L = \text{line}$

Then $I_p(X, L) = \text{mult}_p(f|_L)$

Pf. Change coords so $p = [0:0:1]$

& $L = Z(y)$

Let $\bar{f}(x) = f(x, 0, 1)$

$$I_p(X,L) = \dim \mathcal{O}_{\mathbb{P}^2,[0:0:1]} / (f,y)_{[0:0:1]}$$

$$= \dim \mathcal{O}_{\mathbb{A}^2,(0,0)} / (f,y)_{(0,0)}$$

$$= \dim \left(k[x,y] / (f,y) \right)_{(0,0)}$$

$$= \dim \left(k[x] / (\bar{f}) \right)_0 = \text{smallest degree of a term of } \bar{f}.$$

$$= \text{mult}_p(f|_L)$$

example.

$f(x,y,1) = x^3 y + x^2 y^2 + x^2 + x^3$

$\bar{f}(x) = x^2$

Cor 1. $X = Z(f) \subseteq \mathbb{P}^2$, $p \in X$

TFAE
1. $\text{mult}_p(f|_L) > 1$
2. $L \subset T_p X$
3. $I_p(X, L) > 1$

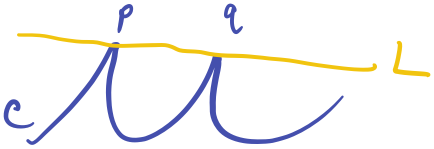
Pf. We already ① $\Leftrightarrow$ ②

Fact gives ① $\Leftrightarrow$ ③

<u>Cor 2</u>. $C \subseteq \mathbb{P}^2$ cubic curve.

Then C has at most 1 sing pt.

<u>Pf</u>. Suppose p, q singular, $p \neq q$.



Let $L = \overline{pq}$ (line)

$T_pC \cong T_qC \cong \mathbb{A}^2$

Cor 1 $\Rightarrow$ $I_p(C, L) \geq 2$

$I_q(C, L) \geq 2$.

Contradicts Bezout $\square$

Higher deg version essentially same:

$\leq \binom{d-1}{2}$ sing pts.

Gathmann Lemma 13.5

Pf of Case 3 Prop

Assume the sing. is at $[0:0:1]$

$\rightsquigarrow f = bx^3 + cx^2y + dxy^2 + ey^3 + q(x,y)$

$q(x,y)$ = quad form in x,y.

(Since $(0,0) \in C$, no const. term.
Since $(0,0)$ sing., no linear terms.)

Have $q(x,y) \not\equiv 0$ because then
f factors into product of 3 linears.

(divide by $y^3 \rightsquigarrow$
poly of deg 3 in x/y ...)

Can factor $q(x,y) = l_0(x,y) l_1(x,y)$ (linear)

Case 1. l_0, l_1 not multiples

Case 2. $l_0 = c l_1$ (multiples).

Clever change of vars.

e.g. in Case 2, WLOG $l_0 = l_1 = y$

$\rightsquigarrow f = bx^3 + cx^2y + dxy^2 + ey^3 + y^2$

Change of vars: $x = x' - \frac{c}{3b} y$

gets rid of x^2y term

etc... $\square$

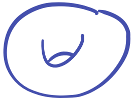
Case 4 Smooth irred cubics.

Prop. C smooth irr cubic

Then C is equiv to some Weierstrass curves.

$$C_{b,c} = Z(f_{b,c})$$

$$f_{b,c} = y^2 - 4x^3 + bx + c$$



Flex pts & Hessians

$p \in C$ is a flex pt (or inflection pt) if $I_p(C, T_pC) \geq 3$

If $C = Z(f) \subseteq \mathbb{P}^2$

$$\rightsquigarrow H_f = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{0 \leq i,j \leq 2}$$

Have: H_f is $\equiv 0$ or homog of deg $3(d-2)$

$\rightsquigarrow$ Hessian curve $H \subseteq \mathbb{P}^2$

Prop. $H \cap C = \{\text{flex pts of } C\}$

Cor. C has a flex pt.

Discriminants

Define $\mathrm{Disc}(f_{b,c})$ to be $b^3 - 27c^2$

Fact. If α_i are roots of f.

$$\mathrm{Disc}(f_{b,c}) = a_n^{2n-2} \prod_{i \neq j} (\alpha_i - \alpha_j)$$

Define $\mathrm{Disc}(C_{b,c}) := \mathrm{Disc}(f_{b,c})/16$

Prop. $C_{b,c}$ smooth $\Leftrightarrow \mathrm{Disc}(C_{b,c}) \neq 0$.

Pf of Prop. Let p = flex pt of C

WLOG $p = [0:0:1]$

& $T_pC = Z(x) = L$

$\rightsquigarrow f|_L$ has 0 of order 3.

at 0.

$\rightsquigarrow f = -y^3 + x(ax^2 + by^2 + cz^2$
$+ dxy + exz + gyz)$

No quadratic terms (flex pt)

Plugging in $x = 0$ needs to give deg 3 in y.

p smooth $\Rightarrow c \neq 0$. Clever change of coords $\square$