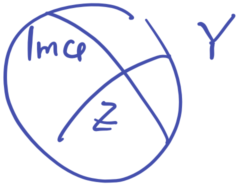
From last time ...

A pos. criterion for dominance

Lemma. $\varphi : X \dashrightarrow Y$ rat'l map b/w proj vars. Y irred. If $\exists$ $Z \subseteq Y$ pav s.t. $\operatorname{im} \varphi$ contains $Y \setminus Z$ then φ is dom.

Pf of Lemma. Follow your nose.

Contradict irreducibility.



Defn of dominant: $\operatorname{im}\varphi$ not contained in proper subvar (assuming Y irred). $\Longleftrightarrow$ dense in Z subsp. top. $\overset{??}{\Longleftrightarrow}$ image open (from last time)

Chap 3 Classical constructions

(Veronese, Segre, Grassmannian).

Veronese Maps

$k[x_0,\dots,x_n]_{(d)} = \left\{ \begin{matrix} \text{deg } d \text{ homog} \\ \text{polys in } x_0,\dots,x_n \end{matrix} \right\}$

$\cong$ k-vect sp on the $\underline{\binom{d+n}{n}}$ monomials of deg d.

#balls #vars - 1

$d+n$ "slots"

choose n slots to put "bars"

$\bullet \; \bullet \; \bullet \; | \; \bullet \; \bullet \; \bullet \; \bullet \; | \; \bullet$

$x_0^3 \, x_1^4 \, x_2^1$

The d-th Veronese map is

$v_d : \mathbb{P}^n \longrightarrow \mathbb{P}^m \qquad m = \binom{d+n}{d} - 1$

$[x_0 : \dots : x_n] \longmapsto [x_0^d : \dots \dots]$

all deg d monomials in $x_0,\dots,x_n$.

- v_d is well def ✓

  (all deg d, don't all vanish)

- v_d is injective.

  look at $x_0^{d-1} x_i$ coords.

  where $x_0 \neq 0$.

  $[x_0^d : x_0^{d-1}x_1 : x_0^{d-1}x_2 : \dots$

  $\sim [x_0 : x_1 : x_2 \dots$

Examples

① $n=1, d=2$

$v_2 : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$

$[s:t] \longmapsto [s^2 : st : t^2]$

$\mathbb{V}_{1,2} = \operatorname{im} v_2 = Z(xz - y^2)$

& v_2 is $\cong$ onto image.

② $n=1, d=3$

$v_3 : \mathbb{P}^1 \longrightarrow \mathbb{P}^{\binom{4}{1}-1} = \mathbb{P}^3$

$[s:t] \longmapsto [s^3 : s^2 t : st^2 : t^3]$

$\mathbb{V}_{1,3} = \operatorname{im} v_3$ = rat'l normal curve of deg 3

= proj clos. of twisted cubic :

$W = Z(xw - yz, y^2 - xz, wy - z^2)$

Easy: $\operatorname{im} v_3 \subseteq W$

Hard: $W \subseteq \operatorname{im} v_3$ (Arrondo)

Chris: maybe direct proof? Proj vars.

③ $n=1, d$

$\operatorname{im} v_d$ = rat'l norm. curve of deg d

= vanish. set of 2×2 dets

$$\begin{pmatrix} z_{0,d} & z_{1,d-1} & \cdots & z_{d-1,1} \\ z_{1,d-1} & z_{2,d-2} & \cdots & z_{d,0} \end{pmatrix}$$

$z_{ij} \longleftrightarrow s^i t^j \quad i+j=d.$

Check: $z_{i,j} \cdot z_{k,l} = z_{i+k,j+l}$

④ Veronese surface

$$v_2 : \mathbb{P}^2 \longrightarrow \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$$

$$[s:t:u] \longmapsto [s^2 : t^2 : u^2 : st : su : tu]$$

Im v_2 is van set for 2×2 minors

of $\begin{pmatrix} z_0 & z_3 & z_4 \\ z_3 & z_1 & z_5 \\ z_4 & z_5 & z_2 \end{pmatrix}$ (rank 1 condition)

For general deg 2:

Im v_2 = van set for 2×2 minors

of $\underbrace{(Z_{i,j})}_{\text{symmetric}}$ $\quad Z_{i,j} \longleftrightarrow X_{i-1} X_{j-1}$

$\cancel{i \le j}$

Image of Veronese

$s^2 : t^2 : st$ $\quad$ $s^2 : st$ $\quad$ $t^2 : st$

Let $W = \operatorname{im} \nu_d \; (= V_{n,d})$ $\quad$ $I = (i_0, \dots, i_n)$

Let $x^I \longleftrightarrow x_0^{i_0} \cdots x_n^{i_n}$ $\quad$ $\sum i_j = d$

Prop. W is vanish. set of

$$\{ x^I x^J - x^K x^L : I + J = K + L \}$$

Q. Can this be written in terms of determinants

A. Yes?

n+1 rows

d+1 cols

check!

Example of proof $n=1$, $d=2$

$U_s(x) = [s^2 : st]$

$U_t(x) = [st : t^2]$

both equal $[s:t]$ on overlap.

Q. Proof of Prop?

Prop. $\nu_d : \mathbb{P}^n \to W$ is $\cong$ onto image.

Pf. Construct inverse.

On each pt of W at least one x_i^d is nonzero.

$\rightsquigarrow U_{x_i}$ cover W

Define $U_{x_i} \longrightarrow \mathbb{P}^n$

$x \longmapsto$ "$x_j x_i^{d-1}$ coords"

put in correct order

as in proof of injectivity.

These agree on overlaps, give inverse to ν_d $\square$

A possible hint for proving the

Prop:

$$\Theta : k[x^I] \longrightarrow k[x_0, \dots, x_n]$$

Show $\ker \Theta$ gen by the $x^I x^J - x^K x^L$.

Hyperssurface Sections

f = nonzero homog poly of deg $d \geq 1$.

$\rightsquigarrow Z(f)$ = hypersurf of deg d.

For X = pav, $Z(f) \cap X$ called a hypersurf. section.

Thm. $X \setminus (Z(f) \cap X)$ is $\cong$ an affine alg var. (if not $\emptyset$)

Application.

$\mathrm{Poly}_n/\sim$ = {polys of deg n with nonzero discriminant}/scale.

is affine.

$\hookrightarrow$ homog: $\prod(x_i - x_j)$

Pf for $d=1$ $Z(f)$ = hyperplane,

WLOG $x_0 = 0$.

Pf of general d Apply V_d

hypersurf $Z(f) \rightsquigarrow$ hyperplane.

apply $d=1$ case. use fact that V_d(proj variety) = proj isomorphic var

(next page).

example. $f = x^2 - 3yz \subseteq \mathbb{P}^2$

This $(x^2) - 3(yz)$ in Veronese coords $\rightsquigarrow$ linear!

Images of varieties

Prop. $X \subseteq \mathbb{P}^n$ pav

$\Rightarrow \nu_d(X)$ is pav any d.

Pf by example (Harris)

$X = Z(x_0^3 + x_1^3 + x_2^3)$ hyperplane under ν_3

multiply by all x_i to get 3 polys of deg 2×2 mult of the d you chose.

$$X = Z(x_0^4 + x_0 x_1^3 + x_0 x_2^3,\ x_1 x_0^3 + x_1^4 + x_1 x_2^3,\ x_2 x_0^3 + x_2 x_1^3 + x_2^4)$$

hyperplane ν_4

quadrics under ν_2

Apply $\nu_2 \rightsquigarrow$ 3 quadratics.

But $\operatorname{Im} \nu_2$ is van set of 6 quadratics.

So im X is van set of 9 quadratics.