Q. Suppose $Z \subseteq \mathbb{P}^n$ dense

& Z = image of a morphism.

on q.p. var

Then Z open?

Optional homework:

① above

② Image of V_d =

$Z(x^I x^J - x^K x^L)$

Segre Map

Goal: products of pav's are pav's.

easy for affine space since
$\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$

Note $\mathbb{P}^m \times \mathbb{P}^n$ not even homeo to $\mathbb{P}^{m+n}$

Identify $\mathbb{P}^{(m+1)(n+1)-1}$ with

$$M_{m+1,n+1}(k) / \text{scalar}.$$

Define $\varphi_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$

$$([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) \longmapsto$$

$$\begin{pmatrix} x_0y_0 & \cdots & x_0y_n \\ \vdots & \ddots & \vdots \\ x_my_0 & \cdots & x_my_n \end{pmatrix} = \begin{pmatrix} x_0 \\ \vdots \\ x_m \end{pmatrix} (y_1 \cdots y_n)$$

"outer product"

$\operatorname{Im} \varphi_{m,n}$ = Segre variety.

Use homog coords

$$z_{ij} \leftrightarrow x_i y_j$$

Example $\varphi_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$$([x_0 : x_1], [y_0 : y_1]) \longmapsto \left[\begin{pmatrix} x_0y_0 & x_0y_1 \\ x_1y_0 & x_1y_1 \end{pmatrix}\right]$$

Note: $\det = 0 \Rightarrow rk \leq 1.$

Also $rk \neq 0 \Rightarrow rk = 1$

Thus $\varphi_{1,1}$ well def &

$$\operatorname{Im} \varphi_{1,1} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \ : \ \det = 0\right\}$$

(lin alg: all rank 1 matrices are outer products)

Claim: $\varphi_{1,1}(\mathbb{P}^1 \times pt)$ is linear

($\leftrightarrow$ plane in k^4)

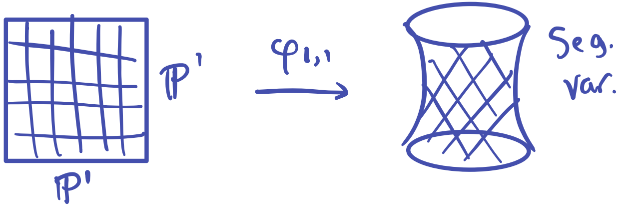
Pf. $pt = [1 : b]$

$$\varphi_{1,1}([x_0 : x_1], [1 : b])$$

$$= \left[\begin{pmatrix} x_0 & bx_0 \\ x_1 & bx_1 \end{pmatrix}\right]$$

$$z_{01} = b\, z_{00}$$
$$z_{11} = b\, z_{10}$$

intersection of 2 3-planes in k^4.



Prop. $\varphi_{m,n}$ injective.

Pf. Let $M = (m_{ij}) = \varphi_{m,n}(a,b)$

WLOG $a_0 = b_0 = 1 \Rightarrow m_{00} = 1$

Recover a, b from first col, row resp. $\square$

Prop. $\operatorname{Im} \varphi_{m,n} = \{\text{rank } 1 \text{ matrices}\}/\text{scale}$.

Pf. Use above lin alg fact or:

Say rk of $M = (m_{ij}) = 1$

Scale so $m_{00} = 1$

$\forall\, k, \ell \neq 0 \quad m_{k\ell} = m_{k0}\, m_{0\ell}$ (i^{th} col is $m_{0i} \cdot 1^{\text{st}}$)

Take a, b to be first col/row.

Algebraic structure on $\mathbb{P}^m \times \mathbb{P}^n$

$\varphi_{m,n}$ gives $\mathbb{P}^m \times \mathbb{P}^n$ an alg. structure:

- Varieties in $\mathbb{P}^m \times \mathbb{P}^n$ are intersections of vars in $\mathbb{P}^N$ with $\operatorname{im} \varphi_{m,n}$ (subspace Zar. topology)
- poly fns on $\mathbb{P}^m \times \mathbb{P}^n$ are poly fns on $\operatorname{im} \varphi_{m,n}$

Prop. Under this defn, subvarieties of $\mathbb{P}^m \times \mathbb{P}^n$ are zero sets of bihomog. polys*.

* if x_i, y_i are coords on $\mathbb{P}^m, \mathbb{P}^n$, each monomial has fixed deg in x_i & fixed deg in y_i. If the deg's are same, say the bihomog. poly is <u>balanced</u>.

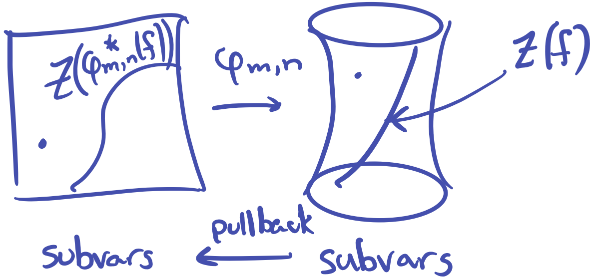
Pf. Given subvar of Segre var:

$$Z(f_1, \dots, f_r)$$

Each f_i pulls back to balanced poly in x, y. If $\deg f_i = d_i$, pullback has bi-degree (d_i, d_i)

e.g. $\varphi_{m,n}^*(z_{00}^2 - z_{01} z_{02})$

$$= (x_0 y_0)^2 - (x_0 y_1)(x_0 y_2)$$



Other direction: Given

$f_1, \ldots, f_r$ bihomog. in x_i, y_i

can make each balanced w/o changing zero set (cf last lecture):

replace f_i with

$$\{y_0 f_i, \ldots, y_n f_i\} \qquad \square$$

Notice: There are many more varieties in $\mathbb{P}^m \times \mathbb{P}^n$ than just products of varieties:

product of vars $\longleftrightarrow$ polys factorable as (poly in x)$\cdot$(poly in y).

- Another way to define products of proj vars:

$$k[X \times Y] = k[X] \otimes k[Y] \ ?$$

(~~Maybe~~ Probably with $k(X)$?)

- $X \times Y$ is a categorical product (satisfies univ property).

Given $\iota_X : Z \to X$

$\iota_Y : Z \to Y$

$\exists$ $\iota : Z \to X \times Y$

s.t. $\pi_X \circ \iota = \iota_X$ same for Y.

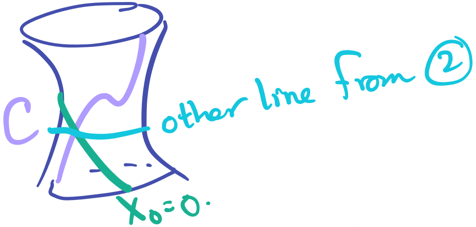
Example. Twisted cubic.

C = image of

$$[s:t] \longmapsto [s^3 : s^2t : st^2 : t^3]$$

Observe $C \subseteq \mathrm{Segre}_{1,1} \subseteq \mathbb{P}^3$.

$$\det\begin{pmatrix} s^3 & s^2t \\ st^2 & t^3 \end{pmatrix} = 0.$$



$\rightsquigarrow C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$.

Besides the eqn defining $\mathrm{Segre}_{1,1}$ there are 2 polys defining C in $\mathbb{P}^3$

① $Z_{00}Z_{10} - Z_{01}^2$

② $Z_{01}Z_{11} - Z_{10}^2$

① Pulls back to C union a line:

$$x_0y_0x_1y_0 - (x_0y_1)^2$$

$$= x_0\underbrace{(y_0^2x_1 - x_0y_1^2)}_{f} \longleftrightarrow \text{line } x_0 = 0 \cup Z(f)$$

Check: $\varphi_{1,1}$ maps $Z(f)$ bij to C.

Coord-free descriptions of ν_d & $\varphi_{m,n}$

$\exists$ natural map

$$k^{n+1} \to \mathrm{Sym}^d(k^{n+1})$$
$$v \longmapsto v^d$$

projectivizing gives ν_d

e.g. $\nu_1 : \mathbb{P}^1 \to \mathbb{P}^2$

$$k^2 \longrightarrow \mathrm{Sym}^2 k^2$$
$$e_1, e_2 \qquad\qquad e_1^2 \quad e_1e_2 \quad e_2^2$$
$$(xe_1+ye_2) \longmapsto (xe_1+ye_2)^2 = x^2(e_1^2) + xy(e_1e_2) + y^2(e_2^2)$$

Similarly $\varphi_{m,n}$ comes from

$$k^{m+1} \times k^{n+1} \longrightarrow (k^{m+1}) \otimes (k^{n+1})$$

$$\mathrm{Sym}^d(V) = V^{\otimes d} / \text{rearranging terms.}$$