Grassmannian

$V = k^n$

$Gr_{r,n} = G_r(V)$

$= \{r\text{-dim subsp's of } V\}$

e.g. $G_{1,n} = \mathbb{P}^{n-1}$

Today: $Gr_{r,n}$ is a proj av.

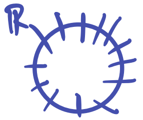
So: The "moduli/parameter space of r-dim lin. varieties is a variety"

Topology aside

B = space.

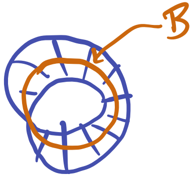
An r-plane bundle over B is a (bigger) space so "over" each $b \in B$, have r-plane.

examples. ① $B = S^1$ $r = 1$



$S^1 \times \mathbb{R}$

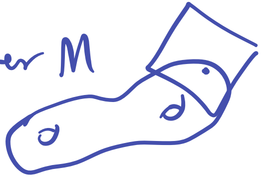
open annulus.



open Mobius band.

② M = smooth n-manifold

TM = n-plane bundle over M



M

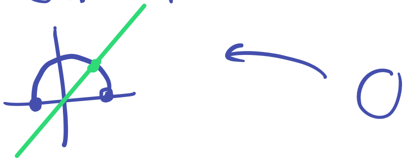
Amazing fact:

$$\left\{ \begin{matrix} r\text{-bundles} \\ \text{over } B \end{matrix} \right\}\Big/\sim \;\longleftrightarrow\; \left\{ B \longrightarrow Gr_{r,\infty} \right\}\Big/\sim$$

Why? $Gr_{r,n}$ (and $Gr_{r,\infty}$) have canonical r-plane bundle E over them.

$$E \subseteq Gr_{r,n} \times k^n$$
$$\| $$
$$\{(W,v) : v \in W\}$$

example. $G_{1,2}$ $k = \mathbb{R}$.

$G_{1,2}$ 

and given $B \to Gr_{r,n}$ can pull back the bundle over $Gr_{r,n}$.

Back to the goal: $Gr_{r,n}$ is pav.

<u>Direct approach</u>

We define

$$Gr_{r,n} \longrightarrow \mathbb{P}^{\binom{n}{r}-1}$$

Given $W \in Gr_{r,n}$

$\rightsquigarrow$ basis $v_1, \dots, v_r$

$\rightsquigarrow$ $r \times n$ matrix

$\rightsquigarrow \left(\binom{n}{r} \text{ minors}\right) \in K^{\binom{n}{r}}$

Different bases give $r \times n$ matrices that differ by mult on left by invertible $r \times r$ matrix A.

This changes all minors by $\det A$.

$\rightsquigarrow$ well def pt in $\mathbb{P}^{\binom{n}{r}-1}$.

Need to show: • injective

• image is variety.

For latter, show the image satisfies Plücker relations:

Denote by $M_{i_1 \cdots i_r}$ the minor...

Given $i_1 < \dots < i_{r-1}$

$j_1 < \dots < j_{r+1}$

$$0 = \sum_{l=1}^{r+1} (-1)^l M_{i_1 \cdots i_{r-1} j_l} M_{j_1 \cdots \hat{j_l} \cdots j_{r+1}}$$

$\rightsquigarrow$ many quadrics

Examples

- $W \in G_{1,3}$ $\quad W = \mathrm{Span}\left\{\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}\right\}$

$\rightsquigarrow (a_0 \;\; a_1 \;\; a_2)$

minors: a_0, a_1, a_2.

- $W \in G_{2,3}$ $\quad W = \mathrm{Span}\{a, b\}$

$\sim \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$

minors $\longleftrightarrow$ cross product.

Plucker: $(-1)\, M_{01} M_{02}$

$i_1 = 0$
$j_1 j_2 j_3 = 012$
$+ M_{02} M_{01}$:(

Can see injectivity in both cases. & surjectivity to $\mathbb{P}^2$

Observation (lon): $G_{1,n} \cong G_{n-1,n}$

$G_{r,n} \cong G_{n-r,n}$

First nontrivial Plucker relation: $G_{2,4}$

$$M_{12} M_{34} - M_{13} M_{24} + M_{14} M_{23}$$

single defining poly.

Second approach: Wedge products

V = vect sp. over k

tensor product

$$V^{\otimes r} = V \times \cdots \times V / \text{multilinearity}.$$

$$= \{\text{finite sums of } v_1 \otimes \cdots \otimes v_r\}$$

subject to

$$(av_1 + a'v_1') \otimes v_2 \otimes v_3$$

$$= a\, v_1 \otimes v_2 \otimes v_3 + a'\, v_1' \otimes v_2 \otimes v_3$$

Why? $\{\text{multilinear maps } V^r \longrightarrow W\}$

$\longleftrightarrow \{\text{linear maps } V^{\otimes r} \longrightarrow W\}$

Next...

$$\Lambda^r V = V^{\otimes r} / \text{alternating}.$$

$$= \{\text{finite sums } v_1 \wedge \cdots \wedge v_r\}$$

subject to multilinearity as above

and: swapping two entries gives -1

So: $v_1 \wedge v_2 \wedge v_3 = -v_2 \wedge v_1 \wedge v_3$

and $v_1 \wedge v_1 \wedge v_2 = -v_1 \wedge v_1 \wedge v_2$

$\Longrightarrow v_1 \wedge v_1 \wedge v_2 = 0$

$(\text{char } k \neq 2)$

Why?

1. $\{$alt. multilin. maps $V^r \to W\}$

$\leftrightarrow$ $\{$lin maps $\Lambda^r V \to W\}$

2. $\Lambda^n K^n \cong K \Longrightarrow$ determinants exist and are unique.

3. Area functions in K^n

$$(e_1+e_2)\wedge e_3 = e_1\wedge e_3 + e_2\wedge e_3$$

area of proj to $(e_1+e_2)e_3$ plane = area of proj to e_1e_3 plane + area of proj to e_2e_3 plane

where e_1+e_2 declared to have length 1

<u>Facts</u> 1. If $v_1,\dots,v_n$ basis for V

then $\{v_{i_1}\wedge\cdots\wedge v_{i_r} : i_1<\cdots<i_r\}$

is a basis for $\Lambda^r V$

$\Longrightarrow \dim \Lambda^r V = \binom{n}{r}$

2. $W \leq V$ subsp of dim r

$T \in \mathrm{Aut}(W)$

$\omega \in \Lambda^r W$

$\Longrightarrow T(\omega) = (\det T)\,\omega$

Plücker embedding

$$F : Gr_{r,n} \longrightarrow \mathbb{P}(\Lambda^r V) \overset{\text{Fact 1}}{=} \mathbb{P}^{\binom{n}{r}-1}$$

$$\mathrm{Span}\{v_1, \dots, v_r\} \longmapsto [v_1 \wedge \cdots \wedge v_r]$$

Well def by fact ②

e.g. $v_1 \wedge v_2 = (v_1 + v_2) \wedge v_2$

$= v_1 \wedge v_2 + \cancel{v_2 \wedge v_2}$

$5 v_1 \wedge v_2 \sim v_1 \wedge v_2$

Todo: · F inj

· $\mathrm{Im}\, F$ is proj. var.

Will do at same time.

Defn. $x \in \Lambda^r V$ is totally decomposable if it's an r-wedge (not a sum)

Note: $\mathrm{Im}\, F = \{\text{totally dec}\}$

$e_1 \wedge e_2 + e_3 \wedge e_4$ is the simplest example of not-(tot. dec)

Lemma. Given nonzero $x \in \Lambda^r V$

Let $\varphi_x : V \longrightarrow \Lambda^{r+1} V$

$v \longmapsto v \wedge x$

① $\dim \ker \varphi_x \leq r$, with $=$ iff x tot. dec.

② If $x = v_1 \wedge \cdots \wedge v_r$ then $\ker \varphi_x = \mathrm{Span}\{v_1, \dots, v_r\}$

Lemma. Given nonzero $x \in \Lambda^r V$.
Let $\varphi_x : V \to \Lambda^{r+1} V$
$v \mapsto v \wedge x$

① $\dim \ker \varphi_x \le r$, with $=$ iff x tot. dec.
② If $x = v_1 \wedge \cdots \wedge v_r$ then $\ker \varphi_x = \mathrm{Span}\{v_1, \ldots, v_r\}$

---

② $\Longrightarrow F$ inj.

second half of ① $\Longrightarrow \mathrm{im} F$ is a variety because:

$x \in \mathrm{im} F \iff x$ tot dec.
$\overset{①}{\iff}$ nullity $\varphi_x \geq r$.
$\iff$ rank $\varphi_x \le n-r$
$\iff$ all $n-r+1$ minors vanish.

$Gr_{r,n} \longrightarrow \Lambda^r V$
$\mathrm{Span}\{v_1, \ldots, v_r\} \longrightarrow v_1 \wedge \cdots \wedge v_r$

$\mathbb{P}(\Lambda^r V) \longrightarrow \mathbb{P}(\mathrm{Hom}_k(V, \Lambda^{r+1} V))$
$x \longmapsto \varphi_x$

inj & linear, can apply $\mathbb{P}$

rank $\le n-r$ defines closed subset of RHS

$\rightsquigarrow$ closed subset of RHS

$\overset{?}{\rightsquigarrow}$ closed subset of $\mathbb{P}(\Lambda^r V)$
(preim. of closed is closed).