Grassmannians

$Gr_{r,n} = \{r\text{-planes in } V = k^n\}$

Goal: this is proj. alg var.

Plücker embedding

$$F : Gr_{r,n} \longrightarrow \mathbb{P}(\Lambda^r V)$$

$$\mathrm{Span}\{v_1, \dots, v_r\} \longrightarrow [v_1 \wedge \cdots \wedge v_r]$$

To show: ① F inj.

② $\mathrm{Im}\, F$ closed.

Note: $\mathrm{Im}\, F = \{\text{tot. dec. elts}\}$

Tool: Wedging map

$x \in \Lambda^r V$

$$\rightsquigarrow \varphi_x : V \longrightarrow \Lambda^{r+1} V$$

$$v \longmapsto v \wedge x$$

Have $\varphi_x \in \mathrm{Hom}_k(V, \Lambda^{r+1} V)$

<u>Lemma</u>. $x \in \Lambda^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$.

$\mathrm{Im}\, F$ closed $\Leftarrow$ ② equality $\iff$ x tot. dec.

F inj $\Leftarrow$ ③ If $x = a \cdot v_1 \wedge \cdots \wedge v_r$ tot dec

$\ker \varphi_x = \mathrm{Span}\{v_1, \dots, v_r\}$

Lemma. $x \in \Lambda^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$.

② equality $\iff$ x tot. dec. $\overset{\text{Given ①}}{\iff}$ $\operatorname{rk} \varphi_x \leq n-r$

③ If $x = a \cdot v_1 \wedge \cdots \wedge v_r$ tot dec

$\ker \varphi_x = \operatorname{Span}\{v_1, \dots, v_r\}$

Proof that ② $\Longrightarrow$ Im F closed:

Have $\Lambda^r V \longrightarrow \operatorname{Hom}_k(V, \Lambda^{r+1} V)$

$x \longmapsto \varphi_x$

injective & linear (check).

↳ uses $r < n$.

So can apply $\mathbb{P}$ ...

$$H : \mathbb{P}(\Lambda^r V) \xrightarrow{\text{linear}} \mathbb{P}\left(\operatorname{Hom}_k(V, \Lambda^{r+1} V)\right)$$

$\uparrow F$

$Gr_{r,n}$

by ② ↗

Image of $Gr_{r,n}$ lies in set W of maps of rank $\leq n-r$. (alg cond)

$$Gr_{r,n} = Z\left(H^*\left(\overset{n-r+1}{\text{minor}} \text{ conditions}\right)\right)$$

$$= H^{-1}(W \cap \operatorname{Im} H)$$

Lemma. $x \in \Lambda^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$.

② equality $\Longleftrightarrow$ x tot. dec

③ If $x = a \cdot v_1 \wedge \cdots \wedge v_r$ tot dec

$\ker \varphi_x = \operatorname{Span}\{v_1, \ldots, v_r\}$

Pf. Choose basis $e_1, \ldots, e_n$ for V

$\rightsquigarrow$ basis e_I for $\Lambda^r V$

Assume $e_1, \ldots, e_s$ is basis for $\ker \varphi_x$

Pf of ① Want $s \leq r$

Say $x = \sum a_I e_I$

Fix some $i \in \{1, \ldots, s\}$

$\varphi_x(e_i) = 0 \Longleftrightarrow a_I = 0$ when $i \notin I$

i.e. every non-0 term of x has an e_i.

Since true for $i \in \{1, \ldots, s\}$

every nonzero term uses $e_1, \ldots, e_s$

$\Longrightarrow s \leq r$

Pf of ② Suppose $s = r$.

Then x is a mult. of $e_1 \wedge \cdots \wedge e_s$

other dir: $x = v_1 \wedge \cdots \wedge v_r$ apply ①

Pf of ③ $\operatorname{Span}\{v_1, \ldots, v_r\} \subseteq \ker \varphi_x$

but dim's same by ②. $\square$

Fact. $x \wedge x = 0 \Longleftrightarrow x$ tot dec.

Local coords on Grassmannian

Consider chart on Im F where

$a_J \neq 0$. WLOG $a_J = a_{1\ldots r}$.
(others differ by permuting coords).

Let $B = r \times n$ matrix of rank r
(row B = pt in $Gr_{r,n}$)

$F(\text{row}(B))$ is

$(b_{11}e_1 + \cdots + b_{1n}e_n) \wedge \cdots \wedge (b_{r1}e_1 \cdots b_{rn}e_n)$

$= \sum a_J e_J$

Only the $b_{ij}e_j$ with $j \in J$ contribute to a_J

Further: a_J is the leftmost minor
$(J = 1\ldots r)$

e.g. $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix}$ $J = 1,2$

$\rightsquigarrow b_{11}b_{22}\, e_1 \wedge e_2 + b_{12}b_{21}\, e_2 \wedge e_1 + \cdots$

$= (b_{11}b_{22} - b_{12}b_{21})\, e_1 \wedge e_2 + \cdots$

So $a_J \neq 0 \iff$ leftmost $r \times r$ matrix is invertible.

$\rightsquigarrow$ can mult. B on left to get

$\left(I_r \;\middle|\; \begin{matrix} b_{1,r+1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{r,r+1} & \cdots & b_{r,n} \end{matrix} \right)$ $\overset{b_{ij}}{\longleftrightarrow}$ copy of $\mathbb{A}^{r(n-r)}$

It's a bijection since RREF unique.

It's also $\cong$ of aff. alg vars.

($\rightarrow$) The a_I are minors.

($\leftarrow$) Need to get b_{ij} as polys in a_I

One example

$$a_{23\cdots rj} = \begin{vmatrix} 0 & 0 & \cdots & 0 & b_{1j} \\ 1 & 0 & & \vdots & \vdots \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & 0 & \\ 0 & 0 & & 1 & b_{rj} \end{vmatrix} = (-1)^{r+1} b_{1j}$$

other cases similar.

The incidence correspondence

$$I = \{(W, v) : W \in Gr_{r,n},\ v \in \mathbb{P}(W)\}$$
$$\subseteq Gr_{r,n} \times \mathbb{P}^{n-1}$$

Thm. I is a proj subvar of $Gr_{r,n} \times \mathbb{P}^{n-1}$

Applications

① $V \subseteq Gr_{r,n}$ proj sub var

$\Longrightarrow \bigcup_{W \in V} W \subseteq \mathbb{P}^{n-1}$ is a subvar.

Pf idea:

$$Gr_{r,n} \xleftarrow{\pi_1} I \xrightarrow{\pi_2} \mathbb{P}^{n-1}$$

$$\bigcup_{W \in V} W = \pi_2 \circ \pi_1^{-1}(V)$$

② $X \subseteq \mathbb{P}^n$ var.

$L_r(X)$ = locus of proj r-planes meeting X.

Prop. $L_r(X)$ is a proj subvar of $Gr_{r+1,n+1}$, hence ⇝ var in $\mathbb{P}^n$ by prev appl.

Pf.

$$\begin{array}{ccc} & I & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ Gr_{r+1,n+1} & & \mathbb{P}^n \end{array}$$

$$L_r(X) = \pi_1 \circ \pi_2^{-1}(X)$$