

Grassmannians

$G_{r,n} = \{r\text{-planes in } V = K^n\}$

Goal: this is proj. alg var.

Plücker embedding

$$F: G_{r,n} \rightarrow \mathbb{P}(\wedge^r V)$$

$$\text{Span}\{v_1, \dots, v_r\} \rightarrow [v_1 \wedge \dots \wedge v_r]$$

To show: ① F inj
② $\text{Im } F$ closed.

Note: $\text{Im } F = \{\text{tot. dec. elts}\}$

Tool: Wedging map

$$x \in \wedge^r V$$

$$\leadsto \varphi_x: V \rightarrow \wedge^{r+1} V$$

$$v \mapsto v \wedge x$$

$$\text{Have } \varphi_x \in \text{Hom}_K(V, \wedge^{r+1} V)$$

Lemma. $x \in \wedge^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$.

$\text{Im } F \text{ closed} \iff$ ② equality $\iff x$ tot. dec.

F inj \iff ③ If $x = a \cdot v_1 \wedge \dots \wedge v_r$ tot dec
 $\ker \varphi_x = \text{Span}\{v_1, \dots, v_r\}$

Lemma. $x \in \wedge^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$. Given ①

② equality $\iff x$ tot. dec. $\iff \text{rk } \varphi_x \leq n-r$

③ If $x = a \cdot v_1 \wedge \dots \wedge v_r$ tot dec
 $\ker \varphi_x = \text{Span}\{v_1, \dots, v_r\}$

$$H: \mathbb{P}(\wedge^r V) \xrightarrow{\text{linear}} \mathbb{P}(\text{Hom}_K(V, \wedge^{r+1} V))$$

$\uparrow F$
 $G_{r,n}$

by ② \nearrow

Image of $G_{r,n}$ lies
in set W of maps of
rank $\leq n-r$. (alg cond)

$$G_{r,n} = Z(H^*(\binom{n-r+1}{\text{minor conditions}}))$$
$$= H^{-1}(W \cap \text{Im } H)$$

Proof that ② $\implies \text{Im } F$ closed:

$$\text{Have } \wedge^r V \rightarrow \text{Hom}_K(V, \wedge^{r+1} V)$$

$$x \mapsto \varphi_x$$

injective & linear (check).

\hookrightarrow uses $r < n$.

So can apply $\mathbb{P} \dots$

Lemma. $x \in \wedge^r V$, $x \neq 0$.

Then ① $\dim \ker \varphi_x \leq r$.

② equality $\iff x$ tot. dec

③ If $x = a \cdot v_1 \wedge \dots \wedge v_r$ tot dec
 $\ker \varphi_x = \text{Span}\{v_1, \dots, v_r\}$

Pf. Choose basis e_1, \dots, e_n for V
 \rightsquigarrow basis e_I for $\wedge^r V$

Assume e_1, \dots, e_s is basis for $\ker \varphi_x$

Pf of ① Want $s \leq r$

Say $x = \sum a_I e_I$

Fix some $i \in \{1, \dots, s\}$

$\varphi_x(e_i) = 0 \iff a_I = 0$ when $i \notin I$

i.e. every non-0 term of x has an e_i .

Since true for $i \in \{1, \dots, s\}$

every nonzero term uses e_1, \dots, e_s
 $\implies s \leq r$

Pf of ② Suppose $s = r$.

Then x is a mult. of $e_1 \wedge \dots \wedge e_s$

other dir: $x = v_1 \wedge \dots \wedge v_r$ apply ③

Pf of ③ $\text{Span}\{v_1, \dots, v_r\} \subseteq \ker \varphi_x$

but dim's same by ②. \square

Fact. $x \wedge x = 0 \iff x$ tot dec.

Local coords on Grassmannian

Consider chart on $\text{Im} F$ where

$a_J \neq 0$. WLOG $a_J = a_1 \dots a_r$.
(others differ by permuting
coords).

Let $B = r \times n$ matrix of rank r
(row $B = pt$ in $Gr_{r,n}$)

$F(\text{row}(B))$ is

$$(b_{11}e_1 + \dots + b_{1n}e_n) \wedge \dots \wedge (b_{r1}e_1 + \dots + b_{rn}e_n) \\ = \sum a_J e_J$$

Only the $b_{ij}e_j$ with $j \in J$ contribute to a_J

Further: a_J is the leftmost minor
($J = 1 \dots r$)

e.g. $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{pmatrix} \quad J = 1, 2$

$$\rightsquigarrow b_{11}b_{22}e_1 \wedge e_2 + b_{12}b_{21}e_2 \wedge e_1 + \dots \\ = (b_{11}b_{22} - b_{12}b_{21})e_1 \wedge e_2 + \dots$$

So $a_J \neq 0 \iff$ leftmost $r \times r$ matrix
is invertible.

\rightsquigarrow can mult. B on left to get
 $\left(I_r \mid \begin{matrix} b_{1,r+1} & \dots & b_{1,n} \\ \vdots & & \vdots \\ b_{r,r+1} & \dots & b_{r,n} \end{matrix} \right) \xleftrightarrow{b_{ij}}$ copy of $A^{r(n-r)}$

It's a bijection since RREF unique.
 It's also \cong of aff. alg vars.

(\rightarrow) The a_I are minors.

(\leftarrow) Need to get b_{ij} as polys
 in a_I

One example

$$a_{23 \dots rj} = \begin{vmatrix} 0 & 0 & & 0 & b_{1j} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & & 0 & b_{rj} \end{vmatrix} = (-1)^{r+1} b_{1j}$$

other cases similar.

The incidence correspondence

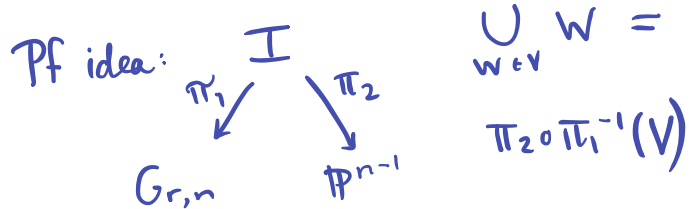
$$I = \{(W, V) : W \in G_{r,n}, V \in \mathbb{P}(W)\} \\ \subseteq G_{r,n} \times \mathbb{P}^{n-1}$$

Thm. I is a proj subvar of $G_{r,n} \times \mathbb{P}^{n-1}$

Applications

(1) $V \subseteq G_{r,n}$ proj subvar

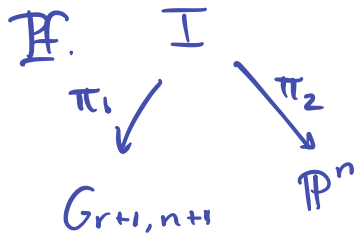
$\Rightarrow \bigcup_{W \in V} W \subseteq \mathbb{P}^{n-1}$ is a subvar.



② $X \subseteq \mathbb{P}^n$ par.

$L_r(X)$ = locus of proj r -planes meeting X .

Prop. $L_r(X)$ is a proj subvar of $G_{r+1, n+1}$, hence \rightsquigarrow par in \mathbb{P}^n by prev appl.



$$L_r(X) = \pi_1 \circ \pi_2^{-1}(X)$$

