Incidence Correspondence

$$I = I_{r,n} = \{(W,v) : v \in \mathbb{P}(W)\}$$
$$\subseteq Gr_{r,n} \times \mathbb{P}^{n-1}$$

$$\mathbb{P}(W) = \{W \setminus 0\}/\text{scale} \subseteq \mathbb{P}^{n-1}$$

Thm. $I_{r,n}$ is proj subvar of $Gr_{r,n} \times \mathbb{P}^{n-1}$

Fact 1. X, Y proj av's
$U \subseteq X$ open $\Rightarrow$ $U \times Y$ open in $X \times Y$

Pf. Suffices closed $\times Y$ is closed.
(↑ zero set of homog. polys in x_i)

Can use same polys as fns on $X \times Y$.

Recall from Segre: Closed sets in $X \times Y$ are van sets of bihomog. poly's

(Topology)
Fact 2. $A \subseteq X$ closed
$\Leftrightarrow$ $\exists$ open cover $\{X_i\}$ of X s.t. $X_i \cap A$ closed in X_i (in subsp top)

Pf ($\Leftarrow$) $X_i \setminus A$ open in X_i hence X
& $X \setminus A$ is union of these.

Fact 3 (Lin Alg)

$A = (I_r \mid B)$ $r \times n$.

Then $v \in \text{Row}\, A \iff \forall i$

$(i^{th} \text{ col of } A) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} = v_i$

Pf. $v \in \text{Row}\, A$

$\iff v \in \text{Col} \begin{pmatrix} I_r \\ B^T \end{pmatrix}$

$\iff \begin{pmatrix} I_r \\ B^T \end{pmatrix} x = v$ consistent

$\iff \begin{pmatrix} I_r \\ B^T \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} = v$ $\square$

Fact 4 $f \in k[x_1, \dots, x_n, y_0, \dots, y_m]$ homog in y's then $Z(f)$ is closed in $\mathbb{A}^n \times \mathbb{P}^m$ in subsp top $\rightsquigarrow$ Pf: homog in x.

Pf of Thm. Cover $Gr_{r,n}$ by open sets $U_{a_{i_1 \dots i_r}}$. By Facts 1+2 suffices to show

$(U_{a_{i_1 \dots i_r}} \times \mathbb{P}^{n-1}) \cap I_{r,n}$ closed in $U_{a_{i_1 \dots i_r}} \times \mathbb{P}^{n-1}$

We'll do $U_{a_{1 \dots r}}$ i.e. subset of $Gr_{r,n}$ given by $\{A = (I_r \mid B)\}$

By Fact 3, the intersection given by

$(i^{th} \text{ col of } B) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} = v_{i+r}$

This poly is homog in v_i $\overset{\text{Fact 4}}{\rightsquigarrow}$ closed subset of $U_{a_{1 \dots r}} \times \mathbb{P}^{n-1}$

$\square$

More variety!

Four constructions

① **Prop.** $V \subseteq Gr_{r,n}$ subvar

$\Longrightarrow X = \bigcup_{W \in V} W$ subvar of $\mathbb{P}^{n-1}$

Pf.

$$\begin{array}{ccc} & I_{r,n} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ V \subseteq Gr_{r,n} & & \mathbb{P}^{n-1} \end{array}$$

$$X = \pi_2 \circ \pi_1^{-1}(V)$$

Need: • π_i are continuous

• π_i are closed.

② $X \subseteq \mathbb{P}^n$ pav.

$L_r(X)$ = locus of proj r-planes meeting X

Prop. $L_r(X)$ subvar of $G_{r+1,n+1} =: Gr_{r,n}$

Pf. $L_r(X) = \pi_1 \circ \pi_2^{-1}(X)$.

③ Joins

$X, Y \subseteq \mathbb{P}^n$ subvars

$J(X,Y) = \{$lines in $\mathbb{P}^n$ meeting both$\}$

Prop. $J(X,Y)$ subvar of $G_{2,n+1} = G_{1,n}$

Pf. $J(X,Y) = L_1(X) \cap L_2(Y)$

④ Fano varieties

$F_r(X) = \{$r-planes contained in $X\}$

$\subseteq Gr_{r,n}$.

Projections are Morphisms

$X \subseteq \mathbb{P}^n, \; Y \subseteq \mathbb{P}^m$

Segre $\rightsquigarrow$ $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$

$$\pi_Y : X \times Y \longrightarrow Y$$
$$(x, y) \longmapsto y$$

Prop. π_Y is a morphism

Pf. Suffices to do

$$\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^m$$

Recall Segre:

$$(x,y) \longmapsto \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} (y_0 \cdots y_m)$$

$$= \begin{pmatrix} x_0 y_0 & \cdots & x_0 y_m \\ \vdots & & \vdots \\ x_n y_0 & \cdots & y_n y_m \end{pmatrix}$$

any nonzero row is the proj to Y.

(need to check agreement on overlap. but all rows are multiples so ✓)

Prop. π_Y is closed.

Note: false in affine case

(project $xy=1$ to $\mathbb{A}^1$, get $\mathbb{A}^1 \setminus 0$)

Thm. $f: X \to Y$ morphism $Z \subseteq X$ subvar. Then $f(Z) \subseteq Y$ subvar. (all proj)

Cor. X connected proj var. Then any (global) regular fn is const.

Pf. $X \xrightarrow{\text{reg fn}} \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ not surj

image is a subvar by Thm

$\Rightarrow$ image is finite set of pts. Done by connected. $\square$

Tool: Graphs

$f: X \to Y$ morphism

$\rightsquigarrow \gamma_f: X \to X \times Y$

$x \mapsto (x, f(x))$

Image Γ_f is graph of f.

Lemma. Γ_f closed in $X \times Y$ & $\gamma_f: X \to \Gamma_f$ is $\cong$.

To prove Thm. Lemma allows us to assume f is $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$