Projections are closed

\[ X \subseteq \mathbb{P}^n, \ Y \subseteq \mathbb{P}^m \]
\[ \sim \rightarrow X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m \]
\[ \pi_Y : X \times Y \rightarrow Y \]

Prop. \( \pi_Y \) closed.

(false for affine! hyperbola!)

More generally...

Thm. Any \( f : X \rightarrow Y \) morphism is closed "compactness property"

Cor. Global reg fn\(s \) const.

if \( X \) conn.

Graphs \( f : X \rightarrow Y \)

\[ \tilde{f}_f : X \rightarrow X \times Y \]
\[ x \mapsto (x, f(x)) \]

\[ \text{Image}(\tilde{f}_f) = \Gamma_f \]

\[ \Gamma_f = \{ (x, y) \in \mathbb{P}^n \times \mathbb{P}^m : f(x_0, ..., x_n) = y_i \ \forall i \} \]

assuming \( f = (f_0, ..., f_m) \) on open set in \( X \)

Prop1. \( \Gamma_f \) closed in \( X \times Y \)

\[ J_f : X \rightarrow \Gamma_f \text{ is } \subseteq. \]
Prop 1. $\Gamma_f$ closed in $X \times Y$

If $f : X \to \Gamma_f$ is $\sim$.

F. Morphism. At any $x \in X$

There open $U$ s.t. $f$ given by

$f_0, \ldots, f_m \in k[x_0, \ldots, x_n]$

same deg. On $U$, post-comp with Segre map gives

$(x_0 \ldots x_n)(f_0 \ldots f_m) \sim x_i f_j$

- this agrees on overlaps
- same deg
- image in $\Gamma_f$
- $y_i f_j$ don't sim. vanish

Closed. Let $(p, q) \not\in \Gamma_f$ i.e. $f(p) \neq q$ Choose $U \subseteq \mathbb{P}^n$ nbd of $p$ so $f$ def on $U$ by $f_0, \ldots, f_m$ of deg $d$.

Let $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ van set of $2 \times 2$ minors of

$(f_0 \ldots f_m)$ e.g. $f_0 y_1 = f_1 y_0$

- bihomog of deg $(d, 1)$
- $(U \times \mathbb{P}^m) \cap Z$ open nbd of $(p, q)$

in $\Gamma_f^c$

exactly

($\Gamma_f$ would be $U \times Z$. Problem is that

$f$ is only def. locally.)

Isomorphism Inverse is projection
Prop. \( \pi: \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^m \)

closed.

"Main thm of elimination theory"

Lemma. \( g_1, \ldots, g_r \in \mathbb{P}(k[x_0, \ldots, x_n]) \)

\( \deg d \)

Regard \( g_i \in \mathbb{P}^N \) (take coeffs)

Let \( D \geq d \). Then

\( \{(g_1, \ldots, g_r) \in \mathbb{P}^N \} \cap \mathbb{P}(k[x_0, \ldots, x_n])_D \)

is closed in \( \mathbb{P}^N \).

\[ N = \binom{n+d}{d} \]

Gathmann

Pf of Lemma The condition

\( k[x_0, \ldots, x_n]_D \subseteq (g_1, \ldots, g_r)_D \)

equiv to

\( k[x_0, \ldots, x_n]_D = (g_1, \ldots, g_r)_D \)

Since \( \{g_1, \ldots, g_r\} = \{ \sum h_i g_i : h_i \in k[x_0, \ldots, x_n] \} \)

\( \Rightarrow \equiv \) equiv to:

\( F_D: (k[x_0, \ldots, x_n]_{D-1})^r \rightarrow k[x_0, \ldots, x_n]_D \)

\( (h_1, \ldots, h_r) \mapsto \sum h_i g_i \)

being surjective, ie has rank \( \dim k[x_0, \ldots, x_n]_D = \binom{n+D}{D} \)

\( \Leftrightarrow \) one of the minors of \( F_D \)

of that dim is not zero. \( \square \)
Prop. \( \pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m \) closed.

If. Take coords \( x_0, \ldots, x_n \) \( y_0, \ldots, y_m \)

Let \( Z \subseteq \mathbb{P}^n \times \mathbb{P}^m \)

Say \( Z = Z(f_1, \ldots, f_r) \)

\( f_i \) of deg \((d, d)\)

Let \( a \in \mathbb{P}^m \)

Let \( g_i = f_i(\cdot, a) \)

\( \in k[x_0, \ldots, x_n] \)

Will show \( \forall \pi(Z) \) open condition.

\[ \begin{align*}
\text{af } \pi(Z) & \iff \not\exists x \in \mathbb{P}^n \text{ s.t. } (x, a) \in Z \\
& \iff Z_p(g_1, \ldots, g_r) = \emptyset \\
& \iff \sqrt{(g_1, \ldots, g_r)} \supseteq (x_0, \ldots, x_n) \\
& \iff \exists d_i \text{ s.t. } x_i \in (g_1, \ldots, g_r) \forall i \\
& \iff k[x_0, \ldots, x_n]_D \subseteq (g_1, \ldots, g_r)_D \text{ some } D \\
& \text{take } D = \Sigma d_i \\
& \text{open condition on coeffs of } g_i \\
& \text{ (lemma)}
\end{align*} \]

The coeffs of \( g_i \) are poly's in \( a \), i.e. coords on \( \mathbb{P}^m \)
Thm. Any \( f : X \to Y \) morphism is closed.

**Pf.** Say \( Z \subseteq X \) closed.

\[ f : X \xrightarrow{\sim} \Gamma_f \quad \text{(Prop 1)} \]

\[ \implies f(Z) \text{ closed in } \Gamma_f, \]

hence in \( \mathbb{P}^n \times \mathbb{P}^m \).

By Prop 2 \( \pi(f(Z)) = f(X) \)

closed in \( \mathbb{P}^m \).

It is contained in \( Y \), hence closed in \( Y \). \qed

Chap 4 Dim, deg, smoothness.

\( V = \text{vect sp.} \)

\[ \dim V = \text{sup } \{ r : I \text{ strictly dec chain of lin subsp} \} \]

\[ V = V_0 \supset V_1 \supset \cdots \supset V_r \]

\( X = \text{top space} \)

Krull dimension is

\[ \dim X = \text{sup } \{ r : I \text{ strictly dec chain of closed irred sets} \} \]

\[ X = X_0 \supset \cdots \supset X_r \]

Example. \( \dim \mathbb{A}^1 = \dim \mathbb{P}^1 = 1 \)