Projections are closed

$X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$

$\rightsquigarrow X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$

$\pi_Y : X \times Y \to Y$

Prop. π_Y closed.

(false for affine! hyperbola!)

More generally...

Thm. Any $f: X \to Y$ morphism is closed

"compactness property"

Cor. Global reg fns const. if X conn.

Graphs $f: X \to Y$

$$\jmath_f : X \to X \times Y$$
$$x \mapsto (x, f(x))$$

$$\text{Image}(\jmath_f) = \Gamma_f$$

$$\Gamma_f = \{(x,y) \in \mathbb{P}^n \times \mathbb{P}^m : f_i(x_0, \dots, x_n) = y_i \ \forall i\}$$

assuming $f = (f_0, \dots, f_m)$ on open set in X

Prop 1.
- Γ_f closed in $X \times Y$
- $\jmath_F : X \to \Gamma_f$ is $\cong$.

Arrondo

Prop 1. • Γ_f closed in $X \times Y$

• $j_F : X \to \Gamma_f$ is $\cong$.

Pf. Morphism. At any $x \in X$ $\exists$ open U s.t. f given by $f_0, \dots, f_m \in k[x_0, \dots, x_n]$ same deg. On U, post-comp with Segre map gives

$$\begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} (f_0 \cdots f_m) \rightsquigarrow x_i f_j$$

- this agrees on overlaps ✓
- same deg ✓
- image in Γ_f ✓
- $y_i f_j$ don't sim. vanish ✓

Closed. Let $(p,q) \notin \Gamma_f$ i.e. $f(p) \neq q$. Choose $U \subseteq \mathbb{P}^n$ nbd of p so f def on U by $f_0, \dots, f_m$ of deg d.

Let $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ van set of 2×2 minors of $\begin{pmatrix} f_0 & \cdots & f_m \\ y_0 & \cdots & y_m \end{pmatrix}$ e.g. $f_0 y_1 = f_1 y_0$

- bihomog of deg $(d,1)$
- $(U \times \mathbb{P}^m) \cap Z^c$ open nbd of (p,q) in Γ_f^c

(Γ_f would be exactly Z. Problem is that f is only def. locally.)

Isomorphism. Inverse is projection □

Prop. $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ closed.

"Main thm of elimination theory"

Lemma. $g_1, \dots, g_r \in \mathbb{P}(k[x_0,\dots,x_n])$ deg d

Regard $g_i \in \mathbb{P}^N$ (take coeffs)

Let $D \geq d$. Then

$\{(g_1,\dots,g_r) \in (\mathbb{P}^N)^r :$

$$k[x_0,\dots,x_n]_D \subseteq (g_1,\dots,g_r)_D$$

(elts of deg D in the ideal)

is ~~closed~~ open in $(\mathbb{P}^N)^r$

$$N = \binom{n+d}{d}$$

Gathmann

Pf of Lemma The condition

$$k[x_0,\dots,x_n]_D \subseteq (g_1,\dots,g_r)_D$$

equiv to

$$k[x_0,\dots,x_n]_D = (g_1,\dots,g_r)_D \quad (*)$$

Since $(g_1,\dots,g_r) = \{\sum h_i g_i : h_i \in k[x_0,\dots,x_n]\}$

$(*)$ equiv to:

$$F_D : (k[x_0,\dots,x_n]_{D-d})^r \to k[x_0,\dots,x_n]_D$$

$$(h_1,\dots,h_r) \mapsto \sum h_i g_i$$

being surjective, ie has rank $\dim k[x_0,\dots,x_n]_D = \binom{n+D}{D}$

$\iff$ one of the minors of F_D of that dim is not zero. $\square$

Prop. $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ closed.

Pf. Take coords $x_0, \dots, x_n$
$y_0, \dots, y_m$

Let $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$

Say $Z = Z(f_1, \dots, f_r)$

f_i of deg (d,d)

Let $a \in \mathbb{P}^m$

Let $g_i = f_i(\cdot, a)$

$\in k[x_0, \dots, x_n]$

Will show $a \notin \pi(Z)$ open condition.

$a \notin \pi(Z) \iff \nexists\, x \in \mathbb{P}^n$ s.t. $(x,a) \in Z$

$\iff Z_p(g_1, \dots, g_r) = \emptyset$.

$\overset{WN}{\iff} \sqrt{(g_1, \dots, g_r)} \supseteq (x_0, \dots, x_n)$

$\iff \exists\, d_i$ s.t. $x_i^{d_i} \in (g_1, \dots, g_r)\ \forall i$.

$\iff k[x_0, \dots, x_n]_D \subseteq (g_1, \dots, g_r)_D$ some D

take $D = \sum d_i$

open condition on coeffs of g_i
(lemma)

The coeffs of g_i are poly's in a, i.e. coords on $\mathbb{P}^m$

$\square$

Thm. Any $f : X \to Y$ morphism is closed

Pf. Say $Z \subseteq X$ closed.

$\gamma_f : X \xrightarrow{\cong} \Gamma_f$ (Prop 1)

$\Rightarrow \gamma_f(Z)$ closed in Γ_f, hence in $\mathbb{P}^n \times \mathbb{P}^m$

By Prop 2 $\pi(\gamma_f(Z)) = f(X)$ closed in $\mathbb{P}^m$

It is contained in Y, hence closed in Y □

Chap 4 Dim, deg, smoothness.

V = vect sp.

$\dim V = \sup\{r : \exists$ strictly dec chain of lin subsp $V = V_0 \supset V_1 \supset \cdots \supset V_r\}$

X = top space

Applies to varieties.

Krull dimension is

$\dim X = \sup\{r : \exists$ strictly dec chain of closed irred sets $X = X_0 \supset \cdots \supset X_r\}$

Example. $\dim \mathbb{A}^1 = \dim \mathbb{P}^1 = 1$