Chap 4. Dim, deg, smoothness.

V = vect sp.

$\dim V = \sup\{r : \exists$ strict dec chain of subsp. $V = V_0 \supset \cdots \supset V_r\}$

X = top sp. Krull dim

$\dim X = \sup\{r : \exists$ strict dec chain of closed irred subsp $X = X_0 \supset \cdots \supset X_r (\neq \emptyset)\}$

& $\dim \emptyset = \infty$.

↖ we said $\emptyset$ not irred.

X = variety

$\dim X$ = krull dim in Zar. top.

Example. $\dim \mathbb{A}^1 = \dim \mathbb{P}^1 = 1$

Facts ① If $X \neq \emptyset$, Hausdorff then $\dim X = 0$

(Hausdorff $\Rightarrow$ only irreds are pts).

② $\dim X = \sup\{\dim X_i : X_i$ irred comp$\}$

③ $Y \subseteq X \Rightarrow \dim Y \leq \dim X$

& strict if no irred comp of (closure of) Y is irred comp of X.

④ X covered by U_i open $\Rightarrow \dim X = \sup \dim U_i$

Cor of ③ : X irred, $\dim X = 0$

$\Rightarrow X = pt.$

Want : $\dim \mathbb{A}^n = n.$

easy : $\geq n.$

Krull dim

A = ring

$\dim A = \sup\{r : \exists$ strict inc. $P_0 \subset \cdots \subset P_r$ of proper prime ideals$\}$

By our dictionary: $\dim X = \dim k[X].$

Prop. $\dim k[x_1, \dots, x_n] = n$

Cor. $\dim \mathbb{A}^n = n$

Cor. $\dim \mathbb{P}^n = n$ by ④

Example. $\dim Gr_{r,n} = r(n-r)$

$\left(I \,\middle|\, \quad \right)$

↑ $r \times (n-r)$

also using ④.

example →

In $k[x,y]$:

$0 \subset (x) \subset (x,y) \left(\subset k[x,y] \right)$

Prop. $\dim k[x_1, \ldots, x_n] = n$

Pf. Induct on n.

$n = 0$ ✓

Gathmann Comm Alg class notes. Ch 11

Inductive step

Say:

$0 = P_0 \subset P_1 \subset \cdots \subset P_m \subset k[x_1, \ldots, x_n]$

WLOG: $P_1 = (f)$ where f monic in x_n

↖ can asm P_1 principal since $k[x_1, \ldots, x_n]$ UFD. In a non-UFD (prime) might not be prime.

Monic in x_n: leading term x_n^d

In quotient $k[x_1, \ldots, x_n]/P_1$

Show

$0 = \overline{P}_1 \subset \cdots \subset \overline{P}_m$ is str. inc. chain of prime ideals. ← shorter by 1

Now use:

$k[x_1, \ldots, x_{n-1}] \to k[x_1, \ldots, x_n]/P_1$

$x_i \longmapsto \overline{x}_i$

pull back $\overline{P}_i$. Get chain of strict inc* prime id's in $k[x_1, \ldots, x_{n-1}]$

Why is preim of $\overline{P}_2$ not 0?

□

Example

$A = k[x,y]/(y^2 - x^3 + x)$

P = nonzero prime in A

$\varphi : k[x] \longrightarrow A$

$x \longmapsto \bar{x}$

Want $\varphi^{-1}(P) \neq 0$.

Subexample. Why is $\varphi^{-1}(y) \neq 0$?

$x - x^3 \longmapsto y^2 \in (y)$.

Next example

$A = k[x,y]/(y^2 - x^3 + xy)$ ← f

Want $\varphi^{-1}(y) \neq 0$.

$f \in k[x][y]$ $\quad \varphi$(const-in-y term) $\in (y)$

$y^2 + (x)y - x^3$

$\varphi(x^3) \in (y)$

Where using monic??

Next goal

$X \subseteq \mathbb{P}^n$ variety

$\dim X =$ the d s.t.
$\exists$ finite map
$X \to \mathbb{P}^d$

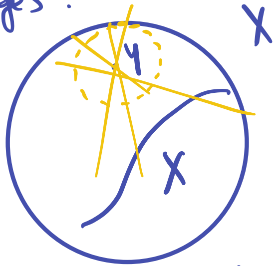
Finite maps

Defn 1. $f: X \to Y$ with dense image
and s.t $f_*: k[Y] \to k[X]$ finite,
meaning $k[X]$ f.g. module
over im f_*

Defn 2. $f: X \to Y$ ~~dense image~~ surj.
& pt preimages are finite.

Why does every variety have
a surj map to $\mathbb{P}^d$ with finite
pt preimages?

Geom answer: $X \subsetneq \mathbb{P}^n$



Stereographic proj $X \to \mathbb{P}^{n-1}$
with finite pt preims:
pt preims are $\mathbb{P}^1 \cap X$ = finite
Can iterate until get surj. map to $\mathbb{P}^d$

Noether normalization

Thm. A = fin gen k-alg

$\Rightarrow \exists\, y_1, \dots, y_d \in A$ alg. indep. s.t.

A is fg as $k[y_1, \dots, y_d]$ module.

On last slide $A = k[X]$.

(Can deduce Nullstullensatz from this.)

Think of y's as transcendental/indep and rest of A as dep. on those.

Example. $A = k[X_1, X_2]/(X_2^2 - X_1^3 + X_1)$

(as above)

$d = 1$, $y_1 = x_1$

x_2 satisfies $f \in k[x_1][Z]$

$$f(Z) = Z^2 - (x_1^3 - x_1)$$

$$\rightsquigarrow A = \{k[x_1] + x_2 k[x_1]\}$$

i.e. A gen by $x_2, 1$ as $k[x_1]$ module.

Notice f is monic in Z.
Can always do lin. change of coords to make it so.
The pf follows then as in example.

Pf of NN in special case:

A gen by one elt c.
(as k-mod)

If c transc. $\Longrightarrow A = k[c]$ done.

If c alg $\Longrightarrow f(c) = 0$ f monic deg d

$\Longrightarrow A = k[z]/(f(z))$

& A gen as a module by $1, c, \dots, c^{d-1}$

$\square$