Chap A. Dim, deg, smoothness.

\[ V = \text{vect sp.} \]
\[ \dim V = \sup \{ r : \exists \text{ strict dec chain of subsp.} \} \]
\[ X = \text{top sp. Krull dim} \]
\[ \dim X = \sup \{ r : \exists \text{ strict dec chain of closed irred subsp} \} \]
\[ X = X_0 \supset \cdots \supset X_r (\neq \emptyset) \]
\[ & \dim \emptyset = \infty. \text{ we said } \emptyset \text{ not irred.} \]

\[ X = \text{variety} \]
\[ \dim X = \text{krull dim in Zar. top.} \]

Example. \( \dim \mathbb{A}^1 = \dim \mathbb{P}^1 = 1 \)

Facts
\[ 1 \] If \( X \neq \emptyset \), Hausdorff
\[ \quad \quad \text{then } \dim X = 0 \]
\[ \text{(Hausdorff } \Rightarrow \text{ only irreds are pts).} \]
\[ 2 \] \( \dim X = \sup \{ \dim X_i : X_i \text{ irred comp} \} \)
\[ 3 \] \( Y \subseteq X \Rightarrow \dim Y \leq \dim X \)
\[ \quad \quad \text{& strict if no irred comp of} \]
\[ \quad \quad \text{(closure of) } Y \text{ is irred comp of } X. \]
\[ 4 \] \( X \) covered by \( U_i \text{ open} \)
\[ \quad \quad \Rightarrow \dim X = \sup \dim U_i \]
Cor. \( \text{of } \circ \): \( X \) irreducible, \( \dim X = 0 \) \( \Rightarrow \) \( X = \text{pt.} \)

Want: \( \dim A^n = n \).

Easy: \( \geq n \).

Krull dim

\[ A = \text{ring} \]

\( \dim A = \sup \{ r : \exists \text{ strict inc. } P_0 \subset \ldots \subset P_r \text{ of proper prime ideals} \} \)

By our dictionary: \( \dim X = \dim k[X] \).

Prop. \( \dim k[x_1, \ldots, x_n] = n \)

Cor. \( \dim A^n = n \)

Cor. \( \dim \mathbb{P}^n = n \) by 4.

Example. \( \dim Gr_n = r(\binom{n}{r}) \)

\[
\begin{bmatrix}
I \\
\end{bmatrix}
\]

\( \leftarrow \uparrow \)

\( r \times (n-r) \)

also using 4.

\[ \ln k[x,y]; \quad 0 < (x) < (x,y) \subset k[x,y] \]
Prop. \( \dim k[x_1, \ldots, x_n] = n \)

If. Induct on \( n \).

\( n = 0 \) ✓

Inductive step

Say:

\( O = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_m \supseteq k[x_1, \ldots, x_n] \)

WLOG: \( P_i = (f) \) where \( f \) monic in \( x_n \)

\( \overset{\uparrow}{\text{can asm } P_i \text{ principal}} \)

since \( k[x_1, \ldots, x_n] \) UFD.

In a non-UFD (prime) might not be prime.

Monic in \( x_n \): leading term \( x_n^d \)

In quotient \( k[x_1, \ldots, x_n]/p \), show

\[ 0 = \overline{P_1} \supseteq \cdots \supseteq \overline{P_m} \]

is str. inc. chain of prime ideals.

Now use:

\[ k[x_1, \ldots, x_{n-1}] \to k[x_1, \ldots, x_n]/p, \]

\( x_i \mapsto \overline{x_i} \)

pull back \( \overline{P_i} \). Get chain of prime id\(_s\) in \( k[x_1, \ldots, x_{n-1}] \)

Why is preim of \( \overline{P_2} \) not 0? ∎
Example

\[ A = k[x, y]/(y^2 - x^3 + x) \]

\[ \text{nonzero} \quad P = \text{prime in} \ A \]

\( \varphi : k[x] \rightarrow A \)

\[ x \mapsto x \]

Want \( \varphi^{-1}(P) \neq 0 \).

Subexample. Why is \( \varphi^{-1}(y) \neq 0 \)?

\[ x - x^3 \mapsto y^2 \in (y) \]

Next example

\[ A = k[x, y]/(y^2 - x^3 + xy) \]

Want \( \varphi^{-1}(y) \neq 0 \).

\[ f \in k[x][y] \]

\[ \varphi (\text{const-in-y term}) \epsilon (y) \]

\[ y^2 + x) y - x^3 \]

\( \varphi(x^3) \epsilon (y) \)

Where using monotonic??
Next goal

\[ X \subseteq \mathbb{P}^n \text{ variety} \]

\[ \dim X = \text{the } d \text{ s.t.} \]

\[ \exists \text{ finite map} \]

\[ X \rightarrow \mathbb{P}^d \]

Finite maps

Defn 1. \( f : X \rightarrow Y \) with dense image and s.t. \( f_* : k[Y] \rightarrow k[X] \) finite,

meaning \( k[X] \) f.g. module over im \( f_* \)

Defn 2. \( f : X \rightarrow Y \) dense image & pt preimages are finite.

Why does every variety have a map to \( \mathbb{P}^d \) with finite pt preimages?

Geom answer:

Stereographic proj \( X \rightarrow \mathbb{P}^{n-1} \)

with finite pt preims:

pt preims are \( \mathbb{P}^{n} \cap X = \text{finite} \)

Can iterate until get surj map to \( \mathbb{P}^d \)
Noether normalization

Thm. $A = \text{fin gen } k$-alg

$\Rightarrow \exists y_1, \ldots, y_d \in A$ \text{ indep. s.t.}

$A$ is fg as $k[y_1, \ldots, y_d]$ module.

On last slide $A = k[X].$

(Can deduce Nullstellensatz from this.)

Think of $y$'s as transcendental/indep

and rest of $A$ as dep. on those.

Example. $A = k[x_1, x_2]/(x_2^2 - x_1^3 + x_1) \quad (\text{as above})$

$d = 1$, $y_1 = x_1$

$x_2$ satisfies $f \in k[x_1][z]$

$f(z) = z^2 - (x_1^3 - x_1) \quad \rightsquigarrow \quad A = \{k[x_1] + x_2 k[x_1]\}$

i.e. $A$ gen by $x_2, 1$ as

$k[x_1]$ module.

Notice $F$ is monic in $Z$. Can always do lin. change of coords to make it so.
The pf follows then as in example.
PF of \( NN \) in special case:

A gen by one elt \( c \).
(as \( k \)-mod)

If \( c \) transc. \( \Rightarrow A = k[c] \) done.

If \( c \) alg \( \Rightarrow f(c) = 0 \) \( \Rightarrow \) \( f \) monic \( \Rightarrow A = k[z]/\langle f(z) \rangle \)

\& A gen as a module

by \( 1, c, \ldots, c^{d-1} \)