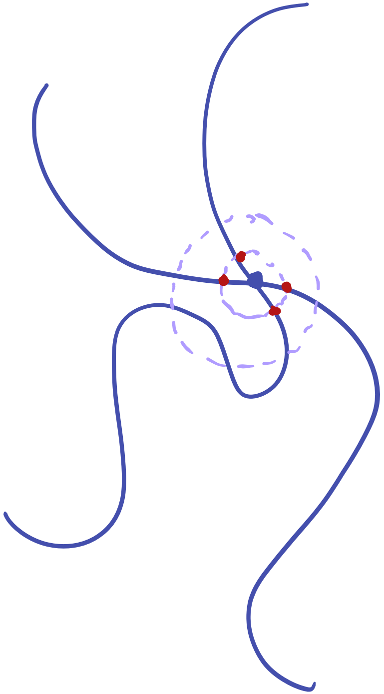
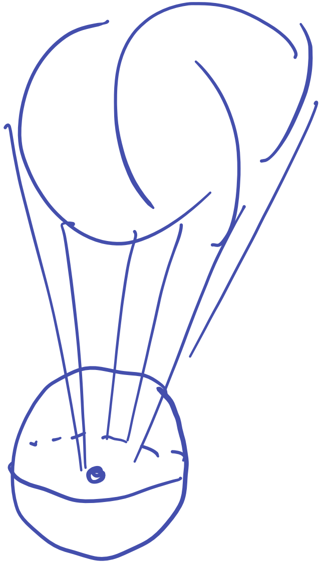
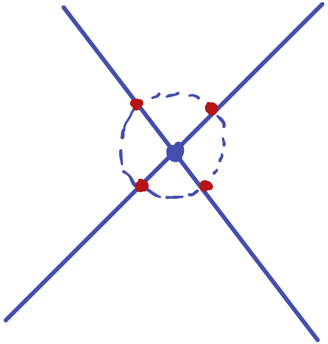
HW due Mon







Hilbert Basis Thm

Thm. Every $Z(I)$ equals some $Z(f_1, \dots, f_r)$

i.e. every aav is the intersection of finitely many hypersurfaces

Lemma/Defn. R ring TFAE

① Every ideal in R is fin gen.

② R satisfies asc. chain cond: any $I_1 \subseteq I_2 \subseteq \dots$ eventually stationary.

Say R is Noetherian.

Fact. Fields are Noetherian.

Pf of Lemma.

① $\Rightarrow$ ② Let $I_1 \subseteq I_2 \subseteq \dots$

$\rightsquigarrow I = \bigcup I_i$ is an ideal.

I is f.g. by ①.

Some I_j contains all gens

so $I_k = I$. $k \geq j$.

② $\Rightarrow$ ① If I not f.g.

make $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$

by adding on gen. at a time.

Prop. R Noetherian $\Rightarrow$ $R[x_1, \dots, x_n]$ Noeth.

In our case $R = K$, so HBT follows.

Pf. We'll do $R[x]$, rest is induction.

Say $I \subseteq R[x]$ not f.g.

Let f_0 = non-0 elt of I of min deg.

Given f_i: f_{i+1} = nonzero elt of $I \setminus \underbrace{(f_0, \dots, f_i)}_{J_i}$ of min deg.

Note $\deg f_i \leq \deg f_{i+1}$

Let a_i = lead coeff of f_i.

$$I_i = (a_0, \dots, a_i) \subset R.$$

R Noeth $\Rightarrow$ $I_0 \subseteq I_1 \subseteq \dots$ eventually stat.

So $\exists$ m st $a_{m+1} \in (a_0, \dots, a_m)$

$$\Rightarrow a_{m+1} = \sum r_i a_i \quad r_i \in R$$

Let $f = f_{m+1} - \sum_{i=0}^{m} x^{\deg f_{m+1} - \deg f_i} r_i f_i$

This f cooked up so $\deg f < \deg f_{m+1}$

Thus $f \in J_m \Rightarrow f_{m+1} \in J_m$

contrad.

$\square$

Hilbert's Nullstullensatz c.1900

Weak Nullst. k alg closed

Every max. ideal in $k[x_1,\dots,x_n]$ is of form $(x_1-a_1,\dots,x_n-a_n)$.

Strong Nullst. k alg closed

$I \subseteq k[x_1,\dots,x_n]$ ideal. Then

$$\mathbb{I}(Z(I)) = \sqrt{I}$$

i.e. $\left\{\begin{matrix}\text{aavs} \\ \text{in } \mathbb{A}^n\end{matrix}\right\} \xleftrightarrow{\text{bij}} \left\{\begin{matrix}\text{rad. ideals} \\ \text{in } k[x_1,\dots,x_n]\end{matrix}\right\}$

$$X \longmapsto \mathbb{I}(X)$$

$$Z(I) \longmapsfrom I$$

The WN implies other natural statements:

- Every proper ideal in $k[x_1,\dots,x_n]$ has a common zero.

  i.e. $I \subsetneq k[x_1,\dots,x_n] \Longrightarrow Z(I) \neq \emptyset$

- Converse: a family of polynomials with no common zeros generates whole $k[x_1,\dots,x_n]$.

Aside: SN is a generalization of Fund Thm Alg.

First, note

$(f) \in \mathbb{C}[z]$ radical

$\iff f$ has no rep. roots.

SN $\Rightarrow$ FTA because

$I(Z(f)) = \sqrt{(f)}$ implies

f has a root.

FTA $\Rightarrow$ SN because

f factors into linears

$\Rightarrow I(Z(f)) = \sqrt{(f)}$

Pf by example:

$$f(z) = (x-1)(x-3)^2$$

$$I(Z(f)) = I(\{1,3\}) = ((z-1)(z-3))$$

$$= \sqrt{(f)}.$$

Both WN & SN fail for k not alg. closed:

e.g. (x^2+1) radical in $\mathbb{R}[x]$

since $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$

But $\mathbb{I}(Z(x^2+1)) = \mathbb{I}(\emptyset) = \mathbb{R}[x]$.

Pf of WN $\Rightarrow$ SN

"Trick of Rabinowitz"

Say $g \in \mathbb{I}(Z(f_1, \dots, f_m))$

Want $g^{\text{some power}} \in (f_1, \dots, f_m)$.

The assumption $\Rightarrow$ a common zero of the f_i is a zero of g.

Thus $f_1, \dots, f_m, x_{n+1}g - 1$ have no common zeros in $\mathbb{A}^{n+1}$

WN $\Rightarrow$ $(f_1, \dots, f_m, x_{n+1}g - 1)$
$= k[x_1, \dots, x_{n+1}]$

$\Rightarrow 1 = p_1 f_1 + \dots + p_m f_m + p_{m+1}(x_{n+1}g - 1)$

where $p_i \in k[x_1, \dots, x_{n+1}]$

Apply the map

$$k[x_1, \dots, x_{n+1}] \longrightarrow k(x_1, \dots, x_n)$$
$$x_i \longmapsto x_i$$
$$x_{n+1} \longmapsto 1/g$$

$\rightsquigarrow 1 = p_1(x_1, \dots, x_n, \frac{1}{g}) f_1 + \dots + p_m(x_1, \dots, x_n, 1/g) f_m$ in $k(x_1, \dots, x_n)$

$$= \frac{\text{Something in } (f_1, \dots, f_m)}{g^{\text{power}}}$$

$\square$

Fact. Each $(x_1 - a_1, \ldots, x_n - a_n)$ is maximal.

Pf. $k[x_1, \ldots, x_n] / (x_1 - a_1, \ldots, x_n - a_n) \longrightarrow k$

$$f \longmapsto f(a_1, \ldots, a_n)$$

$$1 \longleftarrow\!\!\mapsto 1$$

This is $\cong$ so done.

Thm. k = field, K extension

If K is fin gen as a k-alg

then K is algebraic over k.

Pf of WN. Say m = max ideal in $R = k[x_1, \ldots, x_n]$

$\Rightarrow R/m$ is a field, fin gen as k-alg. (since R is).

Have $k \cap m = \{0\}$. (else $m = R$)

$\rightarrow$ image $\bar{k}$ of k in R/m is $\cong k$.

Thm $\rightarrow R/m$ alg. ext. of $\bar{k}$.

k alg closed $\Rightarrow R/m = \bar{k}$

Under $R \to R/m$

each $x_i \longmapsto \bar{a}_i \in \bar{k}$

Some $\bar{a}_i$ image of $a_i \in k$.

$$\Rightarrow m \supseteq \underbrace{(x_1 - a_1, \ldots, x_n - a_n)}_{m'}$$

But m' maximal

$\Rightarrow m = m'$ $\square$