Morphisms

$X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ aav's

$f: X \to Y$ is a <u>morphism</u>

if it's restriction of a polyn. map $\mathbb{A}^n \to \mathbb{A}^m$.

i.e. $\exists\ f_1, \dots, f_m \in k[x_1, \dots, x_n]$

s.t. $f(x) = (f_1(x), \dots, f_m(x))$

$\forall\ x \in X$.

A morphism is an isomorphism if it has an inverse morphism

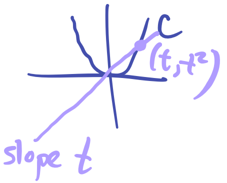
Examples

① Affine change of coords $\mathbb{A}^n \to \mathbb{A}^n$

linear map + translation.

This is $\cong \iff$ linear map is.

② $C = Z(y - x^2) \subseteq \mathbb{A}^2$



$f: \mathbb{A}^1 \to C$

$t \mapsto (t, t^2)$

$f^{-1}: C \to \mathbb{A}^1$

$(x, y) \mapsto x$

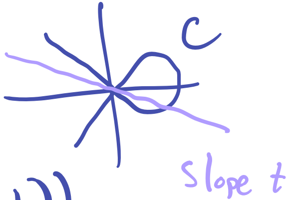
In general, coord fns are morphisms

<u>isomorphism</u>

③ $C = Z(x^3+y^2-x^2)$

$f: \mathbb{A}^1 \longrightarrow C$

$t \longmapsto (t^2-1, t(t^2-1))$



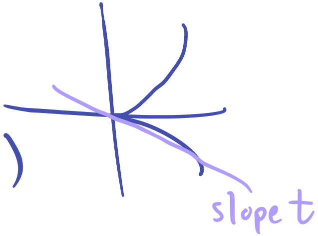
morphism, but not injective:

$f(1) = f(-1)$.

④ $C = Z(y^2-x^3) \subseteq \mathbb{A}^2$

$f: \mathbb{A}^1 \longrightarrow C$

$t \longmapsto (t^2, t^3)$



bijective morphism,

but not $\cong$. Why? We need a new tool...

Facts about morphisms

① Morphisms are continuous wrt Zariski topology

$f: X \to Y$

$f^{-1}(Z(h_1,\dots,h_r))$

$= Z(h_1 \circ f, \dots, h_r \circ f)$

② Morphisms do not always map aav's to aav's

$Z(xy-1) \longrightarrow \mathbb{A}^1$

$(x,y) \longmapsto x$

Image is $\mathbb{A}^1 \setminus 0$

Coordinate Rings

X = aav

$\rightsquigarrow k[X] = \{f|_X : f \in k[x_1, \dots, x_n]\}$
$= \{$poly fns on $X\}$
$=$ coord ring on X.

$k[X]$ is a ring, in fact a k-algebra. More:

$$k[X] = k[x_1, \dots, x_n] / \mathbb{I}(X)$$

So if $X = Z(xy-1)$

$[1/x] \in k[X]$ (!)
$\quad \| $
$[y]$

First examples

① $k[\mathbb{A}^n] \cong k[x_1, \dots, x_n]$

② $k[p] \cong k$ (cf proof $(x_1 - a_1, \dots, x_n - a_n)$ maximal)

$f \longmapsto f(p)$

③ $k[X] \cong k^r$

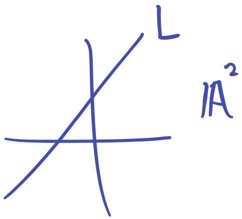
$X = p_1 \cup \dots \cup p_r$

$f \longmapsto (f(p_1), \dots, f(p_r))$

More examples

④ $L = Z(y - mx - b)$

$k[L] \cong k[x]$



First, any poly in x, y is equiv to a poly in x

$y \sim mx + b$

$k[L] \longrightarrow k[x]$

$xy + y \longmapsto x(mx+b) + mx + b$

$[f(x,y)] \longrightarrow f(x, mx+b)$

$k[x] \longrightarrow k[L]$

$f(x) \longmapsto [f(x)]$

These are inverses.

⑤ $C = Z(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$ cone

$k = \mathbb{C}$.

exercise: in $k[C]$

$(x^3 + 2xy^2 - 2xz^2 + x) \sim (x - x^3)$

Irreducibility & Coord rings

Prop. X irred $\iff k[X]$ integral domain

Pf. X irred $\iff \mathbb{I}(X)$ prime

$\iff k[x_1,\dots,x_n]/\mathbb{I}(X)$ integral domain

Fact. $k[X]$ gen. by coord fns,

$$X \to k$$
$$(a_1,\dots,a_n) \mapsto a_i$$

hence the name (?)

⑥ Twisted cubic

$$C = Z(y-x^2, z-x^3)$$

Will show C is irred.

Pf #1 $(y-x^2, z-x^3)$ prime.

Use $k[x,y,z] \cong (k[x,y])[z]$

$k[x,y] \cong (k[x])[y]$

Suppose $fg \in (y-x^2, z-x^3)$.

Division alg:

$f(x,y,z) = (z-x^3)f_1(x,y,z) +$ remainder (const in z)

$(z-x^3)f_1(x,y,z) + (y-x^2)f_2(x,y) + f_3(x)$

similar $g(x,y,z) = (z-x^3)g_1(x,y,z) - (y-x^2)g_2 + g_3(x)$

Since $fg \in (y-x^2, z-x^3) \Rightarrow f_3 = 0$ or $g_3 = 0$

Indeed if $f_3(x)$ & $g_3(x)$ both nonzero, can find a point in $\mathbb{A}^3$ where $y-x^2$, $z-x^3$ vanish but fg does not. ▨

Alternately show:

$$k[x,y,z]\big/(y-x^2, z-x^3) \longrightarrow k[x]$$

<u>Pf #2</u> $f: \mathbb{A}^1 \longrightarrow C$ is a surj. morphism...

$$f \quad t \longmapsto (t, t^2, t^3)$$

If $C = C_1 \cup C_2$ then $\mathbb{A}^1 = f^{-1}(C_1) \cup f^{-1}(C_2)$
$\Longrightarrow \mathbb{A}^1$ irred CONTRAD.

Back to the dictionary

sub-aav's of $X \longleftrightarrow$ rad. ideals in $k[X]$

$$Y \subseteq X \longmapsto k[Y] \subseteq k[X]$$

irred $\longleftrightarrow$ prime

pts $\longleftrightarrow$ max ideals.

<u>Pf.</u> $3^{rd} \cong$ thm + prev. dictionary

<u>Next time</u> Every fin gen., reduced k-alg is some $k[X]$.