Last time

1. Morphisms

$$X \to Y$$

polynomial map

2. Coordinate rings

$k[X]$: poly fns on X

$= \{ f|_X : f \in k[x_1, \dots, x_n] \}$

$= k[x_1, \dots, x_n] / \mathbb{I}(X)$

Wanted to show:

$\mathbb{A}^1 \to \mathbb{Z}(y^2 - x^3)$ not $\cong$.

Next: A morphism $X \to Y$ gives hom. $k[Y] \to k[X]$

Pullbacks. $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ aav's

$f : X \to Y$ morphism.

$\rightsquigarrow f_* : k[Y] \to k[X]$

$g + \mathbb{I}(Y) \mapsto g \circ f + \mathbb{I}(X)$

or $[g] \mapsto [g \circ f]$

Basic facts
1. f_* is k-alg homom.
2. $(fg)_* = g_* f_*$
3. f an $\cong \Rightarrow f_*$ an $\cong$

Basic facts ① f_* is k-alg homom.
② $(fg)_* = g_* f_*$
③ f an $\cong \Rightarrow f_*$ an $\cong$

[In other words, have contravariant functor

$$\begin{array}{ccc} \text{aav's} & \longrightarrow & k\text{-algebras} \\ X & \longmapsto & k[X] \end{array}$$
]

What is the image?

Examples

① $\mathbb{A}^1 \xrightarrow{\cong} Z(y-x^2) \subseteq \mathbb{A}^2$

$t \longmapsto (t, t^2)$

Pullback:

$f: \mathbb{C}[x,y]/(y-x^2) \longrightarrow \mathbb{C}[t]$

$g_1(x,y) = x \longmapsto t$

$g_2(x,y)\ y \longmapsto t^2$

This enough since x, y generate.

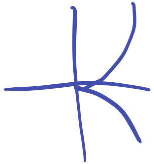
surjective ✓

injective ✓

so $\cong$

② $f: \mathbb{A}^1 \to Z(y^2 - x^3) \subseteq \mathbb{A}^2$

$t \longmapsto (t^2, t^3)$



Pullback:

$$\mathbb{C}[x,y]/(y^2-x^3) \longrightarrow \mathbb{C}[t]$$

$$x \longmapsto t^2$$

$$y \longmapsto t^3$$

Not surj: t not in image

so f is not an $\cong$.

Would be better:

$$\mathbb{C}[x,y]/(y^2-x^3) \not\cong \mathbb{C}[t]$$

Joshua's idea: compare transcendence degree over $\mathbb{C}[x]$.

(Can a transc. ext. of $\mathbb{C}[x]$ be $\cong$ to $\mathbb{C}[x]$?)

More refined: LHS free module over $\mathbb{C}[t]$ of rank 2 and RHS rank 1

③ $X = Z(xy-1)$

Will show $X \not\cong \mathbb{A}^1$

$k[\mathbb{A}^1] = k[x]$

$k[X] = k[x][x^{-1}]$

$= k[x, x^{-1}]$

$=$ Laurent polys

Want $k[x] \not\cong k[x, x^{-1}]$.

Suppose $\Phi : k[x, x^{-1}] \longrightarrow k[x]$

$\Longrightarrow \Phi(1) = 1$

$\Longrightarrow \Phi(x)\Phi(x^{-1}) = 1$

$\Longrightarrow \Phi(x), \Phi(x^{-1})$ units

$\Longrightarrow \operatorname{Im}\Phi \subseteq \{\text{constant polys}\}$

$\blacksquare$

Next: Which alg's arise?

Defn. An alg is <u>reduced</u> if no nilpotent elts, i.e. no elts $r \neq 0$ with $r^k = 0$.

Thm

(1a) Every $k[X]$ is a fin gen. reduced k-alg. ✓

(1b) Every fin gen red. k-alg is a $k[X]$

(2a) $f: X \to Y$ morphism ✓
$\Longrightarrow f_* : k[Y] \to k[X]$ k-alg homom.

(2b) Every k-alg homom $R \to S$ of red fin gen k-alg is some f_* & f unique up to $\cong$.

So. The two categories are same (contravariant isomorphism):

$$\begin{array}{ccc} \text{aav's in } \mathbb{A}^n \text{ over all } n \,/\!\sim & \longleftrightarrow & \text{f.g. red k-algs.} \,/\!\sim \end{array}$$

Note. In 1950's Grothendieck removed 3 hypotheses: fin gen, red, alg closed
The corresp. geom objects are affine schemes

1b R = fg red k-alg

Choose a "presentation"

$$R \cong k[y_1, \ldots, y_m] / J$$

y_i generators

J relations.

$$J = \ker k[y_1, \ldots, y_m] \longrightarrow R$$

R reduced $\Longrightarrow$ J radical.

Let $Y = Z(J) \subseteq \mathbb{A}^m$

SN $\Longrightarrow$ $k[Y] \cong R$. $\square$

2b See 2019 notes.

New varieties from old

① Products

Prop. The direct product of aav's is an aav.

② Complements of aav's

$V \subseteq \mathbb{A}^n$ aav.

$f \in k[V]$

$\rightsquigarrow V_f = V \setminus Z(f)$

$= \{p \in V : f(p) \neq 0\}$

Examples

- $GL_n k$
- $\mathrm{Poly}_n = \{$ polys of deg n with distinct roots $\}$

Prop. Any V_f is isomorphic to an affine variety with coord ring

$$k[V_f] \cong k[V][f^{-1}] = k[V]_f$$

"localization"

rat'l functions

$$\frac{\text{poly}}{f^k}$$

Prop. Any V_f is isomorphic to an affine variety with coord ring

$$k[V_f] \cong k[V][f^{-1}] = k[V]_f$$

Pf. Trick of Rabinowitz!

Let $J = \mathbb{I}(V) \subseteq k[x_1, \dots, x_n]$

$F \in k[x_1, \dots, x_n] \quad F \in [f]$

Set $J_F = (J, tF - 1) \subseteq k[x_1, \dots, x_n, t]$

We'll show $V_f \cong W := Z(J_F) \subseteq \mathbb{A}^{n+1}$

$$W \rightleftarrows V_f$$

$$(x_1, \dots, x_n, y) \longmapsto (x_1, \dots, x_n)$$

$$\left(x_1, \dots, x_n, \frac{1}{F(x_1, \dots, x_n)}\right) \longleftarrow\!\shortmid (x_1, \dots, x_n)$$

Check inverses, check second statement

Example $V = \mathbb{A}^1$

$f = x - 1$