Chap 2 Projective varieties.

Proj space

$\mathbb{P}^n$: compactification of $\mathbb{A}^n$
w/ one infinitely distant
pt in each direction.

$\rightsquigarrow$ compactification of aav's

Precisely:

$$\mathbb{P}^n = (k^{n+1} - 0)/k^*$$
$$= (k^{n+1} - 0) / \text{nonzero scaling}$$
$$= \text{space of lines thru } 0$$

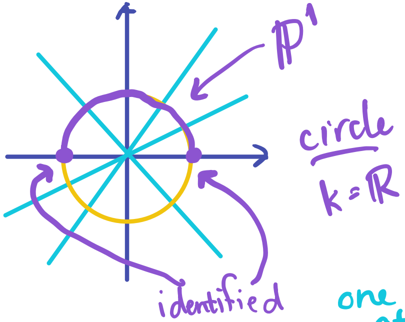
So $x \sim y \iff x = \lambda y \quad \lambda \in k^*$

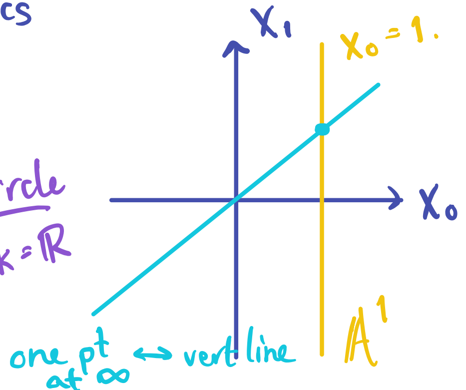
Write
$[(x_0, \ldots, x_n)]$ as $[x_0 : \cdots : x_n]$

"homog. coords"

$\underline{n=1}$ pictures over $\mathbb{R}$

Two pics





algebraically:

$[x_0 : x_1]$

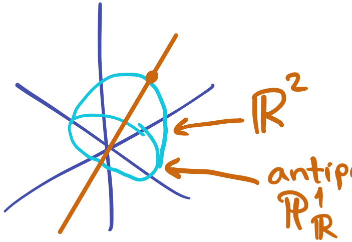
$$\mathbb{P}^1 = \{[1:x_1]\} \cup \{[0:1]\}$$

$$= \quad \mathbb{A}^1 \quad \cup \quad \mathbb{A}^0 = pt$$

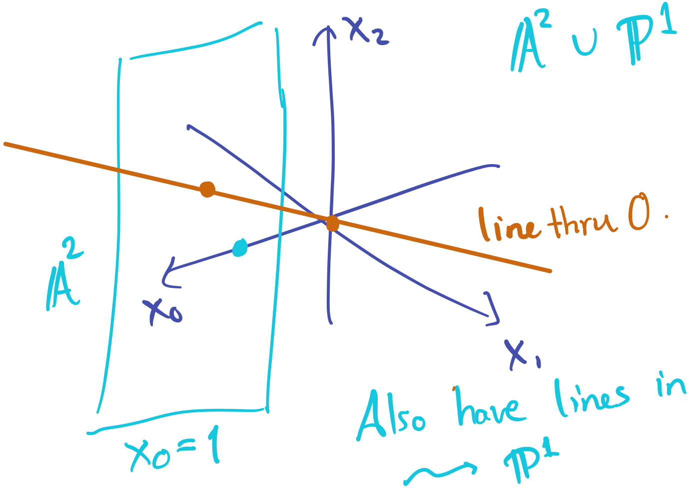
For $k = \mathbb{C}$, $\mathbb{P}^1_{\mathbb{C}}$ = Riemann sphere.

$$= \mathbb{C} \cup \{\infty\}$$

<u>$n=2$</u>



$\mathbb{A}^2 \cup \mathbb{P}^1$



Also have lines in x_1x_2-plane $\rightsquigarrow \mathbb{P}^1$

algebraically:

$$\mathbb{P}^2 = \{[1:x_1:x_2]\} \cup \{[0:x_1:x_2] \;\; x_1, x_2 \text{ not both } 0\}$$

$$\qquad \mathbb{A}^2 \qquad\qquad\qquad \mathbb{P}^1$$

$$= \{[1:x_1:x_2]\} \cup \{[0:1:x_2]\} \cup \{[0:0:1]\}$$

In general:

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$$
$$= \mathbb{A}^n \cup \cdots \cup \mathbb{A}^0$$

This decomp. is not canonical.

Let $U_j = \{[x_0 : \cdots : x_n] : x_j \neq 0\}$

$$\rightsquigarrow \mathbb{P}^n = \underset{\mathbb{A}^n}{\overset{\shortparallel}{U_j}} \cup \underset{\mathbb{P}^{n-1}}{\overset{\shortparallel}{H_j}}$$

The U_j form the <u>standard</u> <u>affine</u> <u>cover</u> of $\mathbb{P}^n$.

For $k = \mathbb{C}$ the U_j give $\mathbb{P}^n$ structure of a $\mathbb{C}$ n-manifold.

<u>Projective subspaces</u>

Images in $\mathbb{P}^n$ of linear subspaces of k^{n+1}.

So a line in $\mathbb{P}^n$ is image of plane in k^{n+1}.

Through any two pts in $\mathbb{P}^n$ $\exists!$ line

<u>Fact.</u> Any two lines in $\mathbb{P}^2$ intersect.

<u>Pf.</u> Any two planes in k^3 intersect.

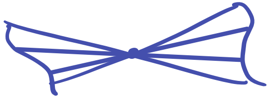
Projective varieties

$f \in k[x_0, \dots, x_n]$ is homog. if all terms have same degree.

Fact. The 0-set of f is well def. in $\mathbb{P}^n$.

Pf. $\lambda^d f(x) = f(\lambda x) = 0$
$\iff f(x) = 0$

Note: $Z(f)$ in $\mathbb{A}^{n+1}$ is a cone



A proj alg var in $\mathbb{P}^n$ is common 0-set of $f_1, \dots, f_r \in k[x_0, \dots, x_n]$ homog.

Examples

(0) $Z(0) = \mathbb{P}^n$

$Z(1) = \emptyset$.

(1) $Z(x_0, \dots, x_n) = \emptyset$.

$(x_0, \dots, x_n) = \{\text{polys w/ no const term}\}$

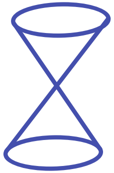
"irrelevant ideal"

(2) $Z(x_1 - a_1 x_0, \dots, x_n - a_n x_0) = [1 : a_1 : \cdots : a_n]$

(3) $Z(x_0)$ = "hyperplane at ∞" $\cong \mathbb{P}^{n-1}$

④ Conics

$X = Z(f)$



e.g. $f = x^2 + y^2 - z^2$

$\exists$ std affine charts

$\rightsquigarrow$ circle, hyperbola, hyperbola

⑤ Image of

$\varphi : \mathbb{P}^1 \to \mathbb{P}^3$

$\varphi([t_0 : t_1]) = [t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3]$

This is a det. variety

$$\operatorname{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \leq 1$$

$\rightsquigarrow$ intersection of 3 quadrics.

" proj. rat'l normal curve of deg 3 "

exercise: Im φ is the whole variety

Tayesh: 2nd, 3rd cols are multiples of 1st.

(6) $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$

$$([x_0 : x_1], [y_0 : y_1]) \longmapsto$$

$$[x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1]$$

$$\operatorname{Im} \varphi = Z(z_0 z_3 - z_1 z_2)$$

"quadric"

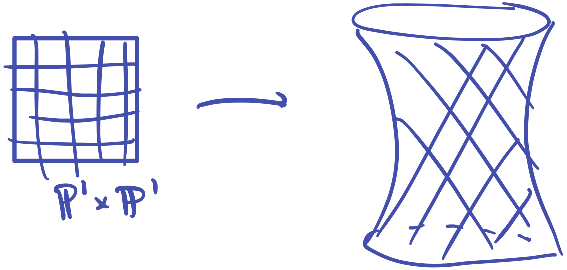


Image of "lines" on left are lines in quadric

e.g. $\varphi(\mathbb{P}^1 \times [1:0])$

$$= Z(z_1, z_3).$$

Q. Do other lines in $\mathbb{P}^1 \times \mathbb{P}^1$ map to lines? (Tong)

Future

(5) Grassmannians

$$G_{r,n} = \{r\text{-dim planes in } k^n\}$$

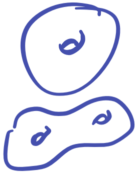
later!

(6) Products of proj. alg. vars.

later!

(7) Compact Riemann surfaces

later?



(8) Moduli space

Homogenization

$f \in k[x_1, \ldots, x_n]$

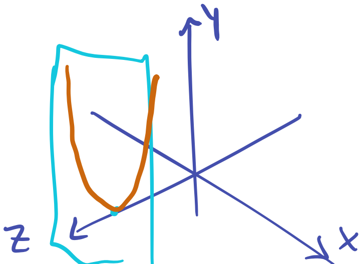
$\rightsquigarrow h \in k[x_0, \ldots, x_n]$

homog.

Just add x_0 as needed.

Example $f(x,y) = y - x^2$

$\rightsquigarrow h(x,y,z) = yz - x^2$



So get the old parabola

plus $[0:1:0]$

$\rightsquigarrow$ parabola + pt $\approx$ circle.

Example ② is also a homogenization.

Upshot: Any affine variety can be projectivized

$\rightsquigarrow$ compactness,

more symmetry.