(Some of) HW assignment.

$$\mathbb{C}^n \xrightarrow{\psi} \mathbb{C}^n$$

roots $\quad$ coeff.

map given by elem sym polys

Surjective: FTA

Not injective: permuting roots.

$$(r_1, \ldots, r_n) \longmapsto \left(\sum_i r_i, \sum_{i \neq j} r_i r_j, \ldots, r_1 \cdots r_n\right)$$

Newton: these generate the invariants.

$$\mathbb{C}^n \downarrow \mathbb{C}^n/\Sigma_n \xrightarrow{\bar{\psi}} \ldots$$

Σ_n: symm gp

$\bar{\psi}$ ← natural one. $\cong$ of varieties.

HW #1. Show X/G is aav. $G \circlearrowright X$ = aav.

via $k[X/G] = k[X]^G$ "invariants"

#2. Show $\bar{\psi}$ is an $\cong$.

Projective closure

$X \subseteq \mathbb{A}^n$ aav.
$\subseteq \mathbb{P}^n$

The proj. closure

$\bar{X}$ is the closure of X in $\mathbb{P}^n$ in Zariski topology

Fact? Same as Eucl. closure.

Write Z_a, $\mathbb{I}_a$
Z_p, $\mathbb{I}_p$
to emphasize affine/proj

Closure: smallest closed set containing...

So proj closure: smallest proj. var containing....
(or largest homog. ideal...)

Easy: Euc. closure $\subseteq$ proj. closure

So: If Euc closure is a pav, it is the proj clos.
Other dir of fact?

Fact. If $X = Z_a(I)$ then
$\bar{X} = Z_p(I_h)$

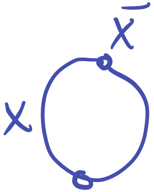
I_h = ideal gen by homog's of all elts of I.

Example.

$X_1 = Z(x_2 - x_1^2)$ $\quad X_2 = Z(x_1 x_2 - 1)$

$\rightsquigarrow \overline{X}_1 = Z(x_0 x_2 - x_1^2)$
$\overline{X}_2 = Z(x_1 x_2 - x_0^2)$ } same!

"Extra" points: Take $x_0 = 0$.



In $\overline{X}_1$: $[0:0:1]$

In $\overline{X}_2$: $[0:0:1]$ & $[0:1:0]$

Not a coincidence: $\exists!$ conic in $\mathbb{P}^2$.

Why is $\overline{X}_i$ actually the proj. closure.

$\overline{X}_i$ is a proj var containg X_i
& $\overline{X}_i \setminus X_i$ finite.

more X_i dense in $\overline{X}_i$

or $\overline{X}_i$ = Euclidean closure of X_i.

note $[1 : x : x^2] \longrightarrow [0:0:1]$
as $|x| \to \infty$.

Fact. If $X = Z_a(f)$ then $\overline{X} = Z_p(f_h)$ homog.

But If $X = Z_a(f, g)$
$\overline{X}$ might not be $Z_p(f_h, g_h)$

Similar: $Z(x, y - x^2)$ [example/exercise: $Z(y - x^2, z - xy)$
$\overline{X} \neq Z(wy - x^2, wz - xy) = \overline{X} \cup \{w = y = 0\}$

Homog. Ideals

Any $f \in k[x_1, \dots, x_n]$ is a sum of homog. terms

$$f = f^{(0)} + \dots + f^{(m)}$$

and "graded ring"

$$k[x_1,\dots,x_n] = \bigoplus_{d \geq 0} \underbrace{k[x_1,\dots,x_n]_{(d)}}$$

homog deg d polys (union 0)

Lemma. Let $I \subseteq k[x_1,\dots,x_n]$.

TFAE ① I gen by homog. elts

② $f \in I \Rightarrow f^{(d)} \in I \;\; \forall d$.

Such I called *homog*.

Pf ② ⇒ ① $I = (f_1, \dots, f_r)$ (Hilbert BT)

Write $f_i = \sum f_i^{(d)} \rightsquigarrow I = (f_i^{(d)})$.

① ⇒ ② $I = (f_1, \dots, f_r)$ each f_i homog.

($r < \infty$ since Noetherian)

$f \in I \Rightarrow f = \sum_i a_i f_i \quad a_i \in k[x_1,\dots,x_n]$

$$\Rightarrow f^{(d)} = \sum_i a_i^{(d - \deg f_i)} f_i \in I \quad \square$$

Note. Not all elts of homog ideals are homog.

Note. A var can always be written as $Z(f_1, \dots, f_k)$ with $\deg f_i$ all same. (mult. each f_i with non-max deg by power of x_0). FIX

Fact. ① I homog $\Rightarrow$ rad I homog.

② Intersection, sum, product of homog. ideals is homog.

③ I homog then:

I prime $\Leftrightarrow$ $\forall$ homog f, g have $(fg \in I \Leftrightarrow f$ or $g \in I)$

(If I homog, can test primeness only with homog elts)

Consequence: Zariski top. works for pav's.

Proj Nullstellensatz

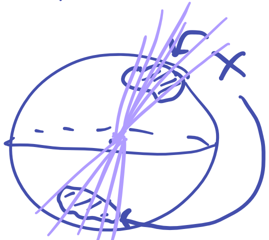
Thm. k alg closed
$I \subseteq k[x_0,\dots,x_n]$ homog.

① $Z_p(I) = \emptyset \iff (x_0,\dots,x_n) \subseteq \operatorname{rad} I$

② $Z_p(I) \neq \emptyset \implies \mathbb{I}_p(Z_p(I)) = \operatorname{rad} I$.

So:
$$\left\{\begin{array}{c}\text{pav's}\\ \text{in } \mathbb{P}^n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{rad. homog. ideals}\\ \text{in } k[x_0,\dots,x_n]\end{array}\right\} \setminus \left\{\begin{array}{c}\text{irrelev.}\\ \text{ideal}\end{array}\right\}$$

Pf uses cones:



$X \subseteq \mathbb{P}^n$

For $X \subseteq \mathbb{P}^n$
cone $C(X)$ is corresp union of lines in k^{n+1}.

Pf of Thm

① $Z_p(I) = \emptyset \iff Z_a(I) \subseteq \{0\}$.
$\iff \operatorname{rad} I = \mathbb{I}_a Z_a(I)$
$\supseteq (x_0,\dots,x_n)$ (affine SN)

② Assume $Z_p(I) \neq \emptyset$.
$f \in \mathbb{I}_p(X) \iff f \in \mathbb{I}_a C(X)$
$= \mathbb{I}_a Z_a(I) = \operatorname{rad} I$

(affine SN)

□